Tree-indiscernibility in SOP_1 and Antichain Trees

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Tree indiscernibility on SOP_1

Definition [DS]

We say T has SOP_1 if there is a formula $\varphi(x, y)$ and tuples $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ such that

(i) for each $\eta \in {}^{\omega}2$, $\{\varphi(x, a_{\eta \restriction n}) | n \in \omega\}$ is consistence,

(ii) for each $\eta, \nu \in {}^{\omega>}2$, $\{\varphi(x, a_{\eta^\frown \langle 1 \rangle}), \varphi(x, a_{\eta^\frown \langle 0 \rangle^\frown \nu})\}$ is inconsistent.

We say T has SOP_2 if there is a formula $\varphi(x, y)$ and tuples $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ such that

(i) for each $\eta \in {}^{\omega}2$, the set $\{\varphi(x, a_{\eta \restriction n}) | n \in \omega\}$ is consistence,

(ii) for each $\eta, \nu \in {}^{\omega>}2$ with $\eta \not\trianglelefteq \nu$ and $\nu \not\trianglelefteq \eta$, the set $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent.

Tree indiscernibility on SOP_1

Definition

For $\bar{\eta} = \langle \eta_0, ..., \eta_n \rangle$, $\bar{\nu} = \langle \nu_0, ..., \nu_n \rangle$ ($\eta_i, \nu_i \in {}^{\omega>}2$), we say $\bar{\eta} \approx_\alpha \bar{\nu}$ if they satisfies

(i) $\bar{\eta}$ and $\bar{\nu}$ are $\wedge$-closed,

(ii) $\eta_i \trianglelefteq \eta_j$ if and only if $\nu_i \trianglelefteq \nu_j$ for all $i, j \leq n$,

(iii) $\eta_i{}^\frown d \trianglelefteq \eta_j$ if and only if $\nu_i{}^\frown d \trianglelefteq \nu_j$ for all $i, j \leq n$ and $d \leq 1$.

We say $\bar{\eta} \approx_\beta \bar{\nu}$ if they satisfy (i), (ii), (iii), and

(iv) $\eta_i{}^\frown\langle 1 \rangle = \eta_j$ if and only if $\nu_i{}^\frown\langle 1 \rangle = \nu_j$ for all $i, j \leq n$.

We say $\bar{\gamma} \approx_\gamma \bar{\nu}$ if they satisfy (i), (ii), (iii), (iv), and

(v) $\eta_i{}^\frown\langle 0 \rangle = \eta_j$ if and only if $\nu_i{}^\frown\langle 0 \rangle = \nu_j$ for all $i, j \leq n$.

(vi) $\eta_i = \sigma{}^\frown\langle 0 \rangle$ for some $\sigma \in {}^{\omega>}2$ if and only if $\nu_i = \tau{}^\frown\langle 0 \rangle$ for some $\tau \in {}^{\omega>}2$, for all $i \leq n$.

(vii) $\eta_i = \sigma{}^\frown\langle 1 \rangle$ for some $\sigma \in {}^{\omega>}2$ if and only if $\nu_i = \tau{}^\frown\langle 1 \rangle$ for some $\tau \in {}^{\omega>}2$, for all $i \leq n$.

We say $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ is α-indiscernible (β, γ-indiscernible, resp.) if $\bar{\eta} \approx_\alpha \bar{\nu}$ ($\bar{\eta} \approx_\beta \bar{\nu}$, $\bar{\eta} \approx_\gamma \bar{\nu}$, resp.) implies $a_{\bar{\eta}} \equiv a_{\bar{\nu}}$.

Tree indiscernibility on SOP_1

Fact [KK] (Modeling Property)

For any $\langle a_\eta \rangle_{\eta \in \omega^{>}2}$, there exists $\langle b_\eta \rangle_{\eta \in \omega^{>}2}$ such that

(i) *$\langle b_\eta \rangle_{\eta \in \omega^{>}2}$ is α-indiscernible,*

(ii) *for any finite set Δ of $\mathcal{L}$-formulas and $\wedge$-closed $\eta = \langle \eta_0, ..., \eta_n \rangle$, there exists $\bar{\nu} \approx_\alpha \bar{\eta}$ such that $\bar{b}_{\bar{\eta}} \equiv_\Delta \bar{a}_{\bar{\nu}}$.*

Lemma

Suppose $\varphi(x, y)$ witnesses SOP_1. Then there is a γ-indiscernible tree $\langle d_\eta \rangle_{\eta \in \omega^{>}2}$ which witnesses SOP_1 with φ.

Theorem

Suppose $\varphi(x, y)$ witnesses SOP_1. Then there is an β-indiscernible tree $\langle e_\eta \rangle_{\eta \in \omega^{>}2}$ which witnesses SOP_1 with φ.

Tree indiscernibility on SOP_1

Notation

(i) For each $\eta \in {}^{\omega>}2$, $l(\eta)$ denotes the domain of η.

(ii) For each $\eta \in {}^{\omega>}2$ with $l(\eta) > 0$, η^- denotes $\eta_{\restriction l(\eta)-1}$ and $t(\eta)$ denotes $\eta(l(\eta) - 1)$.

(iii) For $\bar{\eta} = \langle \eta_0, ..., \eta_n \rangle$, $\mathrm{cl}(\bar{\eta})$ denotes $\langle \eta_0 \wedge \eta_0, ..., \eta_0 \wedge \eta_n \rangle^\frown ... ^\frown \langle \eta_n \wedge \eta_0, ..., \eta_n \wedge \eta_n \rangle$.

(iv) $\eta \perp \nu$ means $\eta \not\trianglelefteq \nu$ and $\nu \not\trianglelefteq \eta$.

Remark

Suppose $\langle \eta_0, ..., \eta_n \rangle \approx_\gamma \langle \nu_0, ..., \nu_n \rangle$. Then it follows that

(i) $\eta_i \wedge \eta_j = \eta_k$ if and only if $\nu_i \wedge \nu_j = \nu_k$ for all $i, j, k \leq n$,

(ii) $\eta_i^- \trianglelefteq \eta_j^-$ if and only if $\nu_i^- \trianglelefteq \nu_j^-$ for all $i, j \leq n$,

(iii) for all $i, j \leq n$, if $\eta_i \perp \eta_j$ then $\eta_i^- \wedge \eta_j^- = \eta_i \wedge \eta_j$,

(iv) $\eta_i ^\frown \langle d \rangle \trianglelefteq \eta_j^-$ if and only if $\nu_i ^\frown \langle d \rangle \trianglelefteq \nu_j^-$ for all $i, j \leq n$ and $d \leq 1$,

(v) $\eta_i^- ^\frown \langle d \rangle \trianglelefteq \eta_j^-$ if and only if $\nu_i^- ^\frown \langle d \rangle \trianglelefteq \nu_j^-$ for all $i, j \leq n$ and $d \leq 1$.

Tree indiscernibility on SOP_1

Sketch of proof (Lemma)

Suppose $\varphi(x, \bar{y})$ witnesses SOP_1 with $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$. For each $\eta \in {}^{\omega>}2$, put $b_\eta := a_{\eta^\frown 0}{}^\frown a_{\eta^\frown 1}$. By the fact, there exists an α-indiscernible $\langle c_\eta \rangle_{\eta \in {}^{\omega>}2}$ such that for any $\bar{\eta}$ and finite subset Δ of $\mathcal{L}$-formulas, $\bar{\nu} \approx_\alpha \bar{\eta}$ and $\bar{b}_{\bar{\nu}} \equiv_\Delta \bar{c}_{\bar{\eta}}$ for some $\bar{\nu}$. For each $\eta \in {}^{\omega>}2 \backslash \{\langle\rangle\}$, we define d'_η by

$$d'_\eta := \begin{cases} c^0_{\eta^-} & \text{if} \quad t(\eta) = 0 \\ c^1_{\eta^-} & \text{if} \quad t(\eta) = 1 \end{cases}$$

and put $d_\eta := d'_{\langle 0 \rangle^\frown \eta}$ for each $\eta \in {}^{\omega>}2$.

Claim 1. φ *witnesses* SOP_1 *with* $\langle d_\eta \rangle_{\eta \in {}^{\omega>}2}$.

Proof of Claim 1. To show this, we define ψ_0 and ψ_1 by

$$\psi_0(x; \bar{y}_0, \bar{y}_1) := \varphi(x, \bar{y}_0) \wedge \bar{y}_1 = \bar{y}_1,$$
$$\psi_1(x; \bar{y}_0, \bar{y}_1) := \varphi(x, \bar{y}_1) \wedge \bar{y}_0 = \bar{y}_0.$$

Tree indiscernibility on SOP_1

Consider the inconsistenty of $\{\varphi(x, d_{\eta^\frown\langle 1\rangle}), \varphi(x, d_{\eta^\frown\langle 0\rangle^\frown\nu})\}$. If $\nu \neq \langle\rangle$, then this is equal to the inconsistency of $\{\psi_1(x, c_\eta), \psi_{t(\nu)}(x, c_{\eta^\frown\langle 0\rangle^\frown\nu^-})\}$. By the modeling property, there exists $\mu, \lambda \in {}^{\omega>}2$ such that

$$\begin{aligned}\langle \mu, \lambda\rangle &\approx_\alpha \langle \langle 0\rangle^\frown\eta, \langle 0\rangle^\frown\eta^\frown\langle 0\rangle^\frown\nu^-\rangle,\\ b_\mu b_\lambda &\equiv_\Delta c_{\langle 0\rangle^\frown\eta} c_{\langle 0\rangle^\frown\eta^\frown\langle 0\rangle^\frown\nu^-},\end{aligned}$$

where Δ contains φ, ψ_0, ψ_1. By $\approx_\alpha$, we have $\mu^\frown\langle 0\rangle \trianglelefteq \lambda$. So $\{\varphi(x, a_{\mu^\frown\langle 1\rangle}), \varphi(x, a_{\lambda^\frown t(\nu)})\}$ is inconsistent. Thus $\{\psi_1(x, b_\mu), \psi_{t(\nu)}(x, b_\lambda)\}$ is inconsistent, $\{\psi_1(x, c_{\langle 0\rangle^\frown\eta}), \psi_{t(\nu)}(x, c_{\langle 0\rangle^\frown\eta^\frown\langle 0\rangle^\frown\nu^-})\}$ is inconsistent, so $\{\varphi(x, d'_{\langle 0\rangle^\frown\eta^\frown\langle 1\rangle}), \varphi(x, d'_{\langle 0\rangle^\frown\eta^\frown\langle 0\rangle^\frown\nu})\}$ is inconsistent, and hence $\{\varphi(x, d_{\eta^\frown\langle 1\rangle}), \varphi(x, d_{\eta^\frown\langle 0\rangle^\frown\nu})\}$ is inconsistent. The case that $\nu = \langle\rangle$ is clear by easy computation. Path consistency can be verified by the same argument. $\square$(Claim 1)

Tree indiscernibility on SOP_1

Claim 2. $\langle d_\eta \rangle_{\eta \in \omega^{>}2}$ *is* γ*-indiscernible.*

Proof of Claim 2. Suppose that $\langle \eta_0, ..., \eta_n \rangle \approx_\gamma \langle \nu_0, ..., \nu_n \rangle$ and for each $i \leq n$, let $\sigma_i = \langle 0 \rangle^\frown \eta_i$ and $\tau_i = \langle 0 \rangle^\frown \nu_i$. By definition of $\langle d_\eta \rangle_{\eta \in \omega^{>}2}$, it is enough to show that $d'_{\sigma_0} ... d'_{\sigma_n} \equiv d'_{\tau_0} ... d'_{\tau_n}$. Clearly $\langle \sigma_0, ..., \sigma_n \rangle \approx_\gamma \langle \tau_0, ..., .\tau_n \rangle$. By using Remark above, one can show that $\mathrm{cl}(\langle \sigma_0^-, ..., \sigma_n^- \rangle) \approx_\alpha \mathrm{cl}(\langle \tau_0^-, ..., \tau_n^- \rangle)$. Since $\langle c_\eta \rangle_{\eta \in \omega^{>}2}$ is α-indiscernible, we have $c_{\sigma_0^-} ... c_{\sigma_n^-} \equiv c_{\tau_0^-} ... c_{\tau_n^-}$. By definition of $\langle d'_\eta \rangle_{\eta \in \omega^{>}2}$,

$$d'_{\sigma_0^- \frown \langle 0 \rangle} d'_{\sigma_0^- \frown \langle 1 \rangle} ... d'_{\sigma_n^- \frown \langle 0 \rangle} d'_{\sigma_n^- \frown \langle 1 \rangle} \equiv d'_{\tau_0^- \frown \langle 0 \rangle} d'_{\tau_0^- \frown \langle 1 \rangle} ... d'_{\tau_n^- \frown \langle 0 \rangle} d'_{\tau_n^- \frown \langle 1 \rangle}.$$

Note that in general, if $m_0 ... m_k \equiv n_0 ... n_k$ and $i_0 < ... < i_e \leq k$, then $m_{i_0} ... m_{i_e} \equiv n_{i_0} ... n_{i_e}$ Since we assume $\langle \eta_0, ..., \eta_n \rangle \approx_\gamma \langle \nu_0, ..., \nu_n \rangle$, we have $t(\sigma_i) = t(\tau_i)$ for each $i \leq n$. Thus

$$d'_{\sigma_0} ... d'_{\sigma_n} \equiv d'_{\tau_0} ... d'_{\tau_n},$$

and this completes proof. $\square$(Claim 2) $\square$(Lemma)

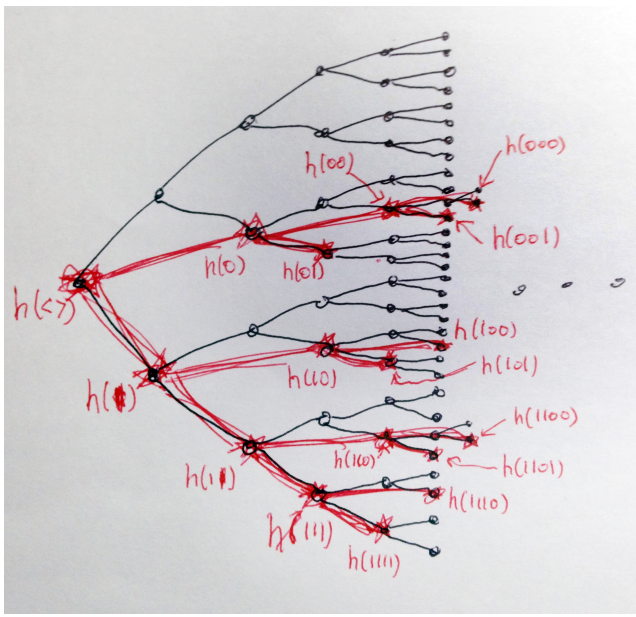
Tree indiscernibility on SOP_1

Sketch of proof (Theorem)

By Lemma, there exists a β-indiscernible tree $\langle d_\eta \rangle_{\eta \in {}^{\omega>}2}$ which witnesses SOP_1 with φ. Define a map $h : {}^{\omega>}2 \to {}^{\omega>}2$ by

$$h(\eta) := \begin{cases} \langle\rangle & \text{if} \quad \eta = \langle\rangle \\ h(\eta^-)^\frown\langle 01\rangle & \text{if} \quad t(\eta) = 0 \\ h(\eta^-)^\frown\langle 1\rangle & \text{if} \quad t(\eta) = 1, \end{cases}$$

and put $e_\eta := d_{h(\eta)}$ for each $\eta \in {}^{\omega>}2$. Then $\langle e_\eta \rangle_{\eta \in {}^{\omega>}2}$ witnesses SOP_1 with φ. And it is β-indiscernible as follows :



Suppose $\bar{\eta} \approx_\beta \bar{\nu}$. Then $h(\bar{\eta}) \approx_\beta h(\bar{\nu})$. The condition (v), (vi), and (vii) are vacuously true so $h(\bar{\eta}) \approx_\gamma h(\bar{\nu})$. Thus $e_{\bar{\eta}} = d_{h(\bar{\eta})} \equiv d_{h(\bar{\nu})} = e_{\bar{\nu}}$. $\square$

Construction of an antichain tree in SOP_1-NSOP_2

Definition

(i) A subset X of ${}^{\omega>}2$ is called an *antichain* if it is pairwisely incomparable (*i.e.* $\eta \perp \nu$ for all $\eta, \nu \in X$).

(ii) By an *antichain tree*, we mean a tuple $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ which satisfies

$$\{\varphi(x, a_\eta) \mid \eta \in X\} \text{ is consistent} \Leftrightarrow X \text{ is an antichain}$$

for all $X \subseteq {}^{\omega>}2$.

Theorem

Suppose $\varphi(x,y)$ *witnesses* SOP_1 *and there is no* $n \in \omega$ *such that* $\bigwedge_{i=0}^{n} \varphi(x, y_i)$ *witnesses* SOP_2. *Then there exists* $\langle b_\eta \rangle_{\eta \in {}^{\omega>}2}$ *such that* $\langle \varphi(x, b_\eta) \rangle_{\eta \in {}^{\omega>}2} \rangle$ *forms an antichain tree.*

Construction of an antichain tree in SOP_1-NSOP_2

Sketch of proof

By the theorem above and compactness, there exists an β-indiscernible tree $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ which witnesses SOP_1 with φ. Define a map $h : {}^{\omega>}2 \to {}^{\omega_1>}2$ by

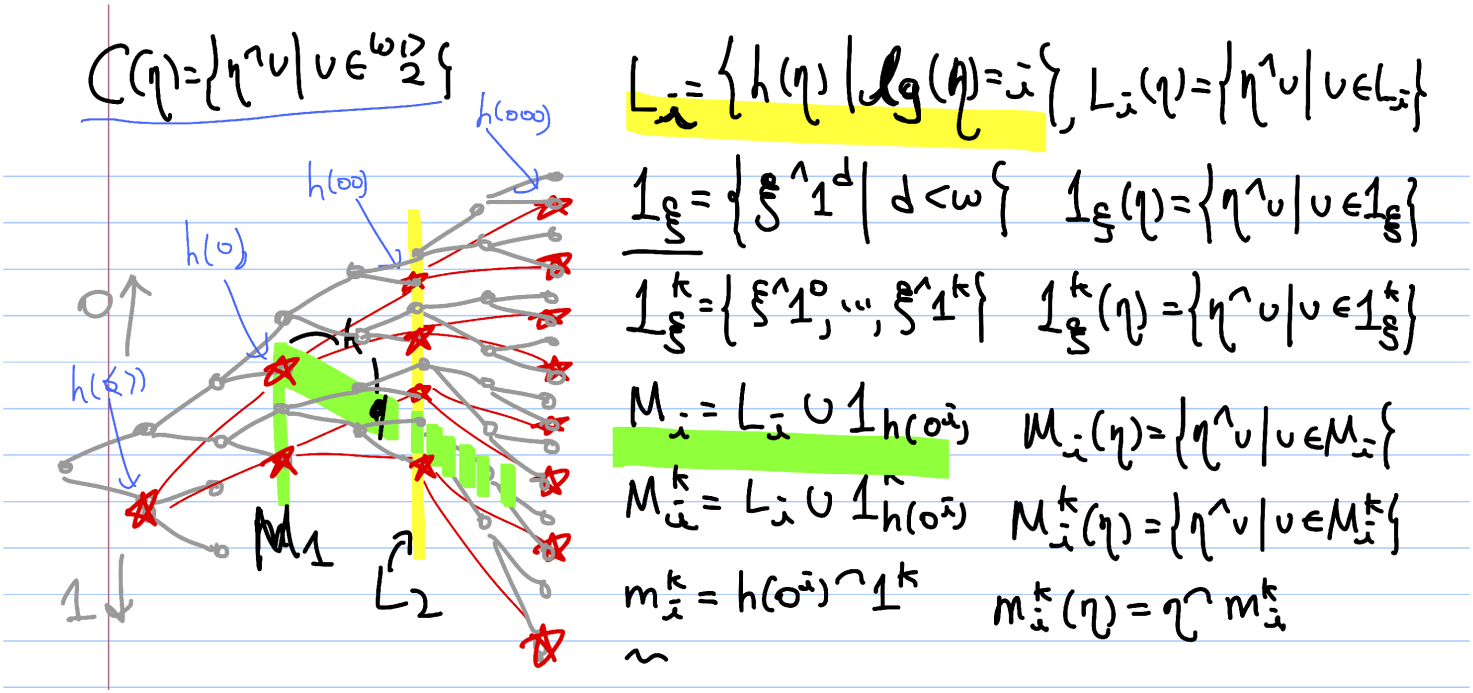
$$h(\eta) := \begin{cases} \langle 1 \rangle & \text{if} \quad \eta = \langle\rangle \\ h(\eta^-)^\frown\langle 001 \rangle & \text{if} \quad t(\eta) = 0 \\ h(\eta^-)^\frown\langle 011 \rangle & \text{if} \quad t(\eta) = 1. \end{cases}$$

For each $i, k \in \omega$ and $\eta \in {}^{\omega_1>}2$, put

$$\begin{aligned} L_i &= \{h(\eta) \mid l(\eta) = i\}, & L_i(\eta) &= \{\eta^\frown\nu \mid \nu \in L_i\} \\ 1_\xi &= \{\xi^\frown\langle 1^d \rangle \mid d \in \omega\}, & 1_\xi(\eta) &= \{\eta^\frown\nu \mid \nu \in 1_\xi\} \\ 1^k_\xi &= \{\xi^\frown\langle 1^0 \rangle, \ldots, \xi^\frown\langle 1^k \rangle\}, & 1^k_\xi(\eta) &= \{\eta^\frown\nu \mid \nu \in 1^k_\xi\} \\ M_i &= L_i \cup 1_{h(\langle 0^i \rangle)}, & M_i(\eta) &= \{\eta^\frown\nu \mid \nu \in M_i\} \\ M^k_i &= L_i \cup 1^k_{h(\langle 0^i \rangle)}, & M^k_i(\eta) &= \{\eta^\frown\nu \mid \nu \in M^k_i\}. \end{aligned}$$

And for each $X \subseteq {}^{\omega_1>}2$, let Φ_X denote $\{\varphi(x, a_\eta) \mid \eta \in X\}$.

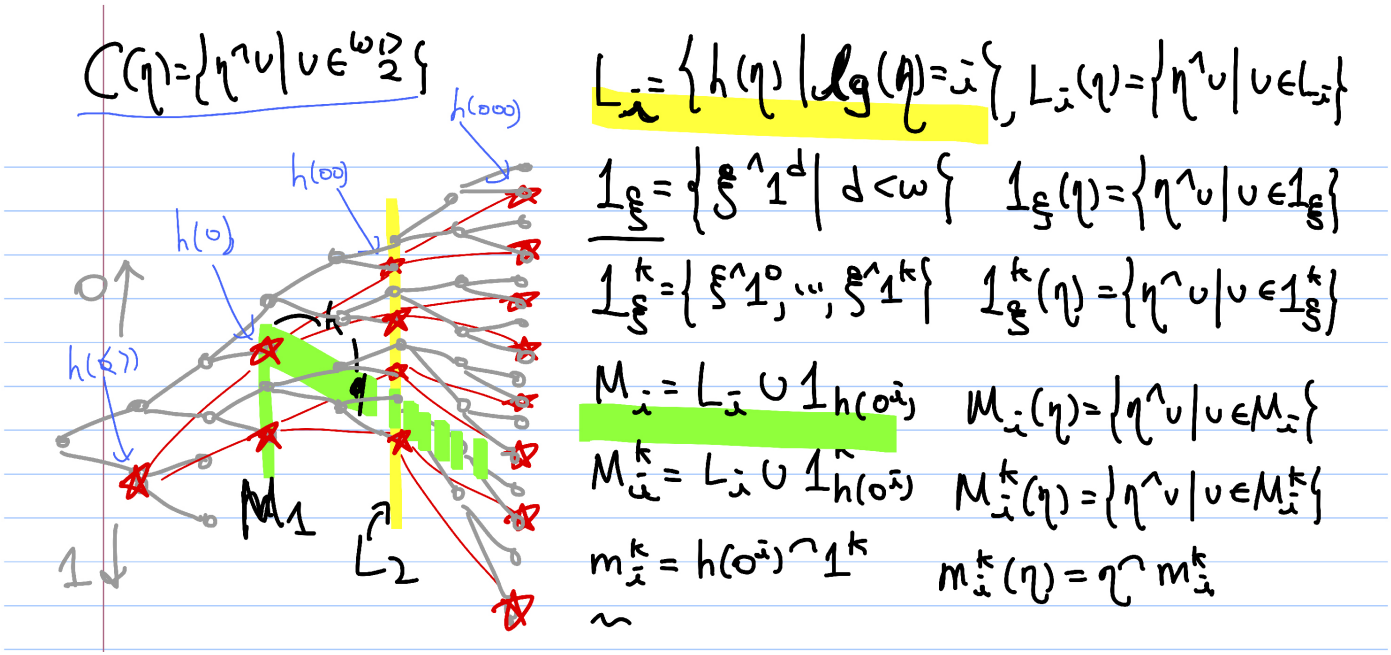
Construction of an antichain tree in SOP_1-NSOP_2



Claim 1. There is $\eta\in {}^{\omega_1>}2$ such that $\Phi_{M_i(\eta)}$ is consistent for all $i\in\omega$.

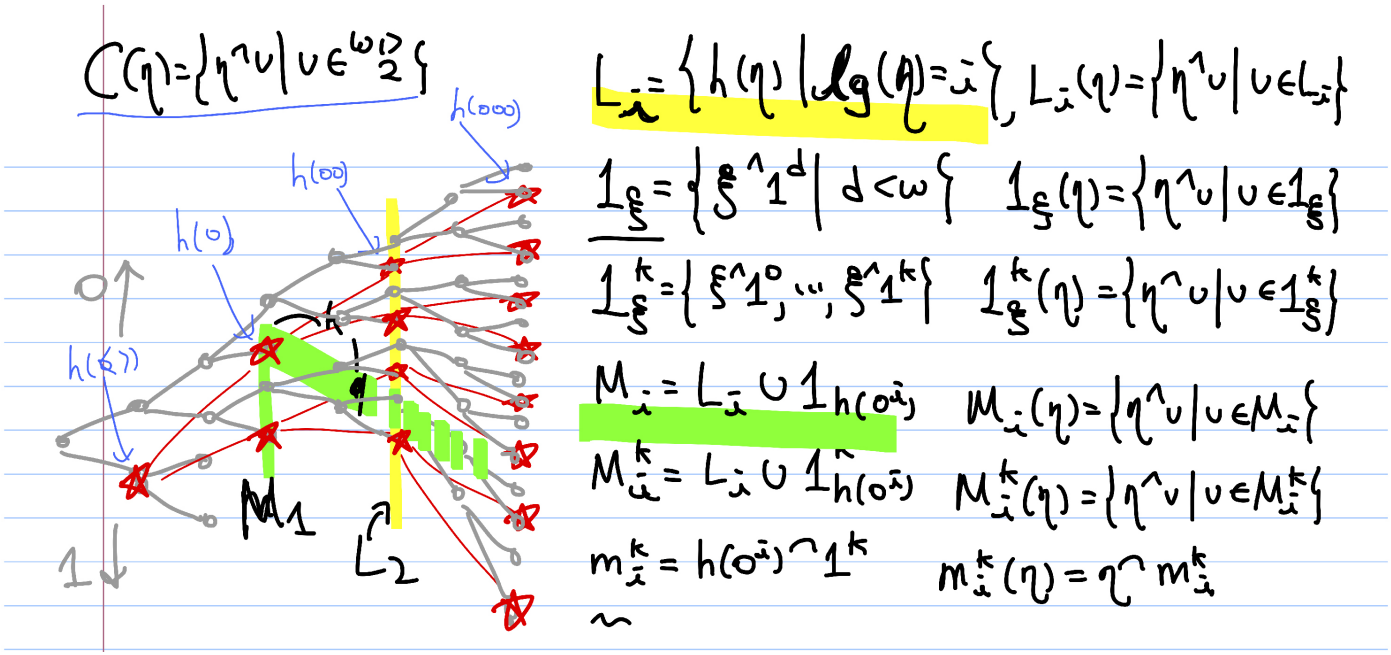
Proof of Claim 1. If not, then we can construct a witness of SOP_2. In fact, if we assume that the claim is not true, then for each $\eta\in {}^{\omega_1>}2$, we can find $i_\eta, k_\eta\in\omega$ such that $\bigcup_{n\in\omega} M_{i_{\eta\restriction n}}^{k_{\eta\restriction n}}$ is consistent for all $\eta\in {}^{\omega}2$, and $M_{i_\eta{}^\frown\langle 0\rangle}^{k_\eta{}^\frown\langle 0\rangle}\cup M_{i_\eta{}^\frown\langle 1\rangle}^{k_\eta{}^\frown\langle 1\rangle}$ is inconsistent with $\bigcup_{n\le l(\eta)} M_{i_{\eta\restriction n}}^{k_{\eta\restriction n}}$ for each $\eta\in {}^{\omega>}2$.

Construction of an antichain tree in SOP_1-$NSOP_2$



By combinatorial way, we may assume there are $i,k\in\omega$ such that $i=i_\eta, k=i_\eta$ for all $\eta\in{}^{\omega>}2$. And we can find a finite set q which satisfies $q\subseteq\bigcup_{n\leq l(\eta)}M_{i_{\eta\restriction n}}^{k_{\eta\restriction n}}$ for all $\eta\in{}^{\omega>}2$, and $M_{i_\eta{}^\frown\langle 0\rangle}^{k_\eta{}^\frown\langle 0\rangle}\cup M_{i_\eta{}^\frown\langle 1\rangle}^{k_\eta{}^\frown\langle 1\rangle}$ is inconsistent with q for each $\eta\in{}^{\omega>}2$. So we can find a finite conjunction $\varphi(x,y_0)\wedge\ldots\wedge\varphi(x,y_n)$ which witnesses SOP_2 with appropriate choice of parameters from $\langle a_\eta\rangle_{\eta\in{}^{\omega>}2}$. But it is impossible because we assumed that $\bigwedge_{i=0}^n\varphi(x,y_i)$ does not witness SOP_2 for all $n\in\omega$. $\square$(Claim 1)

Construction of an antichain tree in SOP_1-NSOP_2

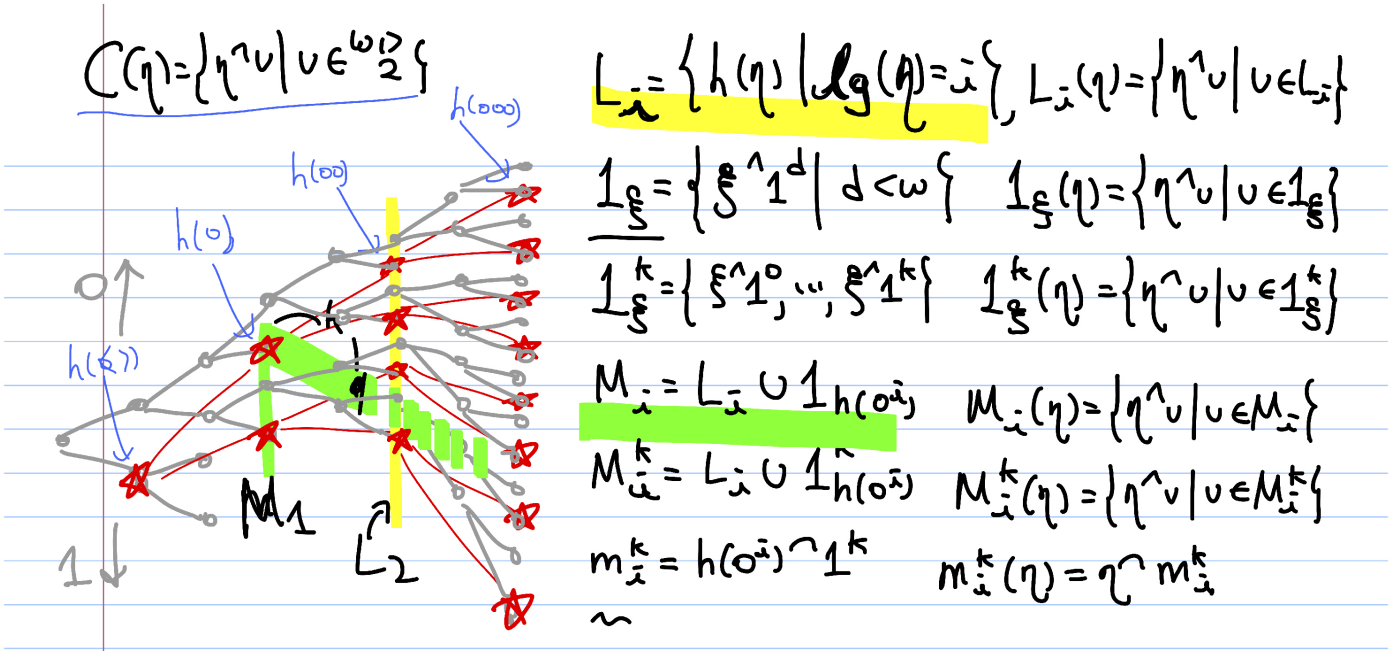


By β-indiscernibility, we may assume that Φ_{M_i} is consistent for all $i\in\omega$. In particular, Φ_{L_i} is consistent for all $i\in\omega$.

Claim 2. $\langle\varphi(x,y),b_\eta\rangle_{\eta\in{}^{\omega>}2}$ forms an antichain tree where $b_\eta=a_{h(\eta)}$ for each $\eta\in{}^{\omega>}2$.

Proof of Claim 2. By the choice of h, it follows that $\eta\trianglelefteq\nu$ implies $(h(\eta)\wedge h(\nu))^\frown\langle 1\rangle=h(\eta)$ and $(h(\eta)\wedge h(\nu))^\frown\langle 0\rangle\trianglelefteq h(\nu)$. So if a subset X of ${}^{\omega>}2$ is not an antichain, then $\{\varphi(x,b_\eta)|\eta\in X\}$ is not consistent.

Construction of an antichain tree in SOP_1-$NSOP_2$



If $X\subseteq {}^{\omega>}2$ is an antichain, then for any finite subset $X_0=\{\eta_0,\dots,\eta_n\}$ of X, there exists $Y_0=\{\nu_0,\dots,\nu_n\}\subseteq {}^{\omega>}2$ such that $\eta_i\trianglelefteq\nu_i$ for all $i\leq n$ and $l(\nu_0)=\dots=l(\nu_n)$. Then $cl(h(X_0))\approx_\beta cl(h(Y_0))$ and in particular $h(X_0)\approx_\beta h(Y_0)$. Since all the elements of Y_0 have the same length, $h(Y_0)\subseteq L_i$ for some $i\in\omega$. Thus $\{\varphi(x,b_\eta)|\eta\in X_0\}$ is consistent and hence $\{\varphi(x,b_\eta)|\eta\in X\}$ is consistence. $\square$(Claim 2) $\square$(Theorem)

Construction of antichain trees

Definition

Let $L_0 = \{<_{lex}, <_{ini}, \cap\}$. For $\bar{\eta}, \bar{\nu} \in {}^{\omega>}2$, we say $\bar{\eta} \approx_{str} \bar{\nu}$ if $\bar{\eta}$ and $\bar{\nu}$ have the same L_0-atomic type (quantifier free type). We say $\langle a_\eta \rangle_{\eta \in \omega>2}$ is str-indiscernible if $\bar{\eta} \approx_{str} \bar{\nu}$ implies $a_{\bar{\eta}} \equiv a_{\bar{\nu}}$.

Fact [TT]

Strong indiscerniblility has modeling property.

Remark

If $\varphi(x, y)$ forms an antichain tree, then there exists a str-indiscernible tree $\langle a_\eta \rangle_{\eta \in \omega>2}$ such that φ forms an antichain tree with $\langle a_\eta \rangle_{\eta \in \omega>2}$, by the modeling property for str-indiscernibility.

Corollary

If T is SOP_1 and $NSOP_2$, then T has an antichain tree $\langle \varphi(x, y), \langle a_\eta \rangle_{\eta \in \omega>2} \rangle$ such that $\langle a_\eta \rangle_{\eta \in \omega>2}$ is str-indiscernible.

properties of antichain trees

Lemma

If $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ *is an antichain tree, then* $\varphi(x,y)$ *witnesses* SOP_1.

Definition

We say T has TP_2 if there is a formula $\varphi(x,y)$ and tuples $\langle a_{i,j} \rangle_{i,j \in \omega}$ and $k \in \omega$ such that

(i) for each $f : \omega \to \omega$, the set $\{\varphi(x, a_{i,f(i)}) | i \in \omega\}$ is consistent,

(ii) for each $i \in \omega$, the set $\{\varphi(x, a_{i,j}) | j \in \omega\}$ is k-inconsistence.

Lemma

If $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ *is an antichain tree, then* $\varphi(x,y)$ *witnesses* TP_2.

In short, the way to prove these lemmas are finding maps $h : {}^{n>}2 \to {}^{\omega>}2$ and $h : n \times n \to {}^{\omega>}2$ such that $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ and $\langle a_{i,j} \rangle_{i,j \in \omega}$ witnesses SOP_1 and TP_2 respectively, where n is arbitrary natural number. But for SOP_2, this method does not work.

properties of antichain trees

Remark

(i) Suppose $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ is an antichain tree. Then there is no $h : {}^{2>}2 \to {}^{\omega>}2$ such that $\langle \varphi(x,y), \langle b_\eta \rangle_{\eta \in {}^{2>}2} \rangle$ satisfies the conditions of SOP_2, where $b_\eta = a_{h(\eta)}$ for each $\eta \in {}^{2>}2$.

(ii) Suppose $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ witnesses SOP_2. Then there is no $h : {}^{2>}2 \to {}^{\omega>}2$ such that $\langle \varphi(x,y), \langle b_\eta \rangle_{\eta \in {}^{2>}2} \rangle$ forms an antichain tree with height 2, where $b_\eta = a_{h(\eta)}$ for each $\eta \in {}^{2>}2$.

Questions

Does β-indiscernibility has modeling property?

Is there an example of $NSOP_2$ theory having an antichain tree?
(If the answer is yes, then $\mathrm{SOP}_1 \neq \mathrm{SOP}_2$.)

Does the existence of antichain tree imply SOP_2?
(If the answer is yes, then $\mathrm{SOP}_1 = \mathrm{SOP}_2$.)

Is there a general way to construct an antichain tree in a $SOP_n \cap NSOP_{n+1}$ theory? ($n \geq 3$)
(If the answer is yes, then SOP_n hierarchy collapse in NTP_2.)

An example of an antichain tree

Let

$\alpha_1 = 1$,

$\alpha_{i+1} = \alpha_i^2 + 1$ for each $i \in \omega$,

β_i : the set of all maximal antichains in ${}^{i>}2$ for each $i \in \omega$,

μ_i : a bijection from α_i to β_i for each $i \in \omega$,

$A_i = \{a_l^i : l < \alpha_i\}$,

$B_i = \{b_\eta^i : \eta \in {}^{i>}2\}$,

$M_i = A_i \cup B_i$.

R : binary relation symbol,

$\mathcal{L} = \{R\}$.

For each $i \in \omega$, let $\mathcal{M}_i$ be an $\mathcal{L}$-structure such that $\mathcal{M}_i = (M_i; R^{\mathcal{M}_i})$, where

$R^{\mathcal{M}_i} = \{(a_l^i, b_\eta^i) \in A_i \times B_i \mid \eta \in \mu_i(l)\}$,

Then $Th(\mathcal{M}_i)$ has an antichain tree with height i, for all $i \in \omega$.

An example of an antichain tree

For each $X \subseteq {}^{\omega>}2$ and $\eta \in {}^{\omega>}2$, let $\eta^\frown X$ denote $\{\eta^\frown \nu : \nu \in X\}$.
For each $i \in \omega$, let $\iota_i : \alpha_i \cup {}^{i>}2 \to \alpha_{i+1} \cup {}^{i+1>}2$ which maps :

$$I \mapsto \mu_{i+1}^{-1}((\langle 0\rangle^\frown \mu_i(I)) \cup \{\langle 1\rangle\}) \qquad \text{for all } I \in \alpha_i,$$

$$\eta \mapsto \langle 0\rangle^\frown \eta \qquad \text{for all } \eta \in {}^{i>}2,$$

and define $\iota_i^* : M_i \to M_{i+1}$ by :

$$a_I^i \mapsto a_{\iota_i(I)}^{i+1},$$

$$b_\eta^i \mapsto b_{\iota_i(\eta)}^{i+1}.$$

Then ι_i^* is an embedding. So we can regard $\mathcal{M}_i$ as a substructure of $\mathcal{M}_{i+1}$ with respect to ι_i^*, for all $i \in \omega$. Put $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$. Then $Th(\mathcal{M})$ has an antichain tree.

An example of an antichain tree

Properties of $Th(\mathcal{M})$

(i) $R(x, y)$ forms an antichain tree.

(ii) $R(x, y)$ does not witness SOP_2.

(iii) $R(y, x)$ does not witness SOP_2.

(iv) $\bigwedge_{i<n} R(x, y_i)$ does not witness SOP_2.

(v) $\bigvee_{i<n} R(x, y_i)$ does not witness SOP_2.

(vi) $\neg R(x, y)$ does not witness SOP_2.

(vii) $\bigwedge_{i<n} \neg R(x, y_i)$ does not witness SOP_2.

But unfortunately, in $Th(\mathcal{M})$, a witness of SOP_2 exists.

(viii) Let $\varphi(x, y) := \neg\exists w(R(w, x) \wedge R(w, y)) \wedge \exists z(x \neq z \neq y \neq x \wedge \exists w(R(w, x) \wedge R(w, z)) \wedge \neg\exists w(R(w, y) \wedge R(w, z)))$. Then φ says "y is a predecessor of x in the set of parameters." (*i.e.*, $y \lhd x$) So, $\varphi(x, y)$ witnesses SOP_2.

References

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