Existence of antichain trees in SOP_1-$NSOP_2$ theories

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Outline

Tree indiscernibility on SOP_1

Construction of an antichain tree in SOP_1-NSOP_2

Properties of antichain trees

Tree indiscernibility on SOP_1

Definition

We say T has SOP_1 if there is a formula $\varphi(x,y)$ and tuples $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ such that

(i) for each $\eta \in {}^{\omega}2$, $\{\varphi(x, a_{\eta\restriction n}) | n \in \omega\}$ is consistence,

(ii) for each $\eta, \nu \in {}^{\omega>}2$, $\{\varphi(x, a_{\eta^\frown\langle 1\rangle}), \varphi(x, a_{\eta^\frown\langle 0\rangle^\frown\nu})\}$ is inconsistent.

We say T has SOP_2 if there is a formula $\varphi(x,y)$ and tuples $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ such that

(i) for each $\eta \in {}^{\omega}2$, the set $\{\varphi(x, a_{\eta\restriction n}) | n \in \omega\}$ is consistence,

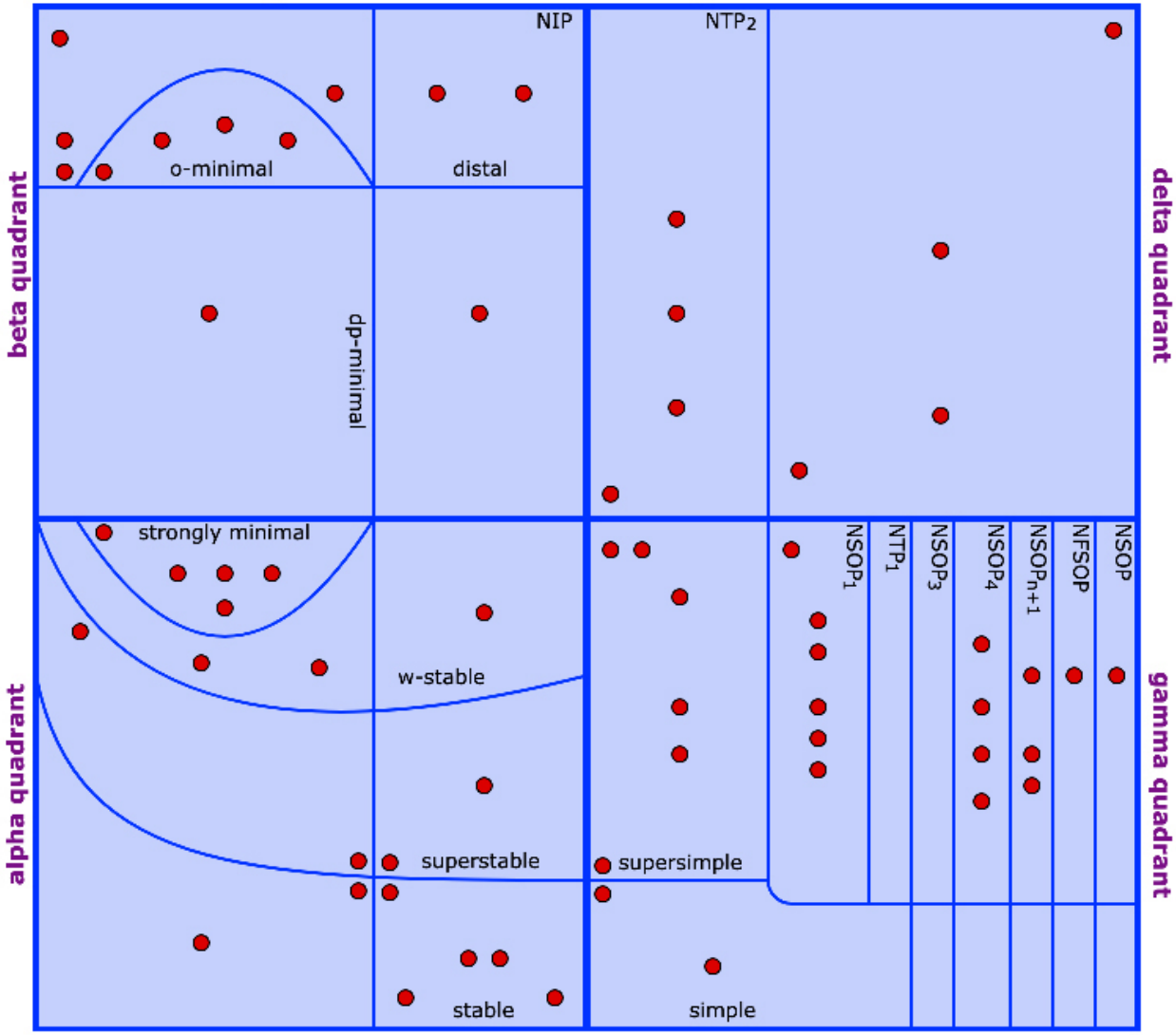
(ii) for each $\eta, \nu \in {}^{\omega>}2$ with $\eta \not\trianglelefteq \nu$ and $\nu \not\trianglelefteq \eta$, the set $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent.

Fact [M. Dzamonja, S. Shelah]

If T has SOP_2, then T has SOP_1.

But we do not know whether the converse holds or not.

forking and dividing



Tree indiscernibility on SOP_1

Definition

For $\bar{\eta} = \langle \eta_0, ..., \eta_n \rangle$, $\bar{\nu} = \langle \nu_0, ..., \nu_n \rangle$ ($\eta_i, \nu_i \in {}^{\omega>}2$), we say $\bar{\eta} \approx_\alpha \bar{\nu}$ if they satisfies

(i) $\bar{\eta}$ and $\bar{\nu}$ are $\wedge$-closed,

(ii) $\eta_i \trianglelefteq \eta_j$ if and only if $\nu_i \trianglelefteq \nu_j$ for all $i, j \leq n$,

(iii) $\eta_i{}^\frown d \trianglelefteq \eta_j$ if and only if $\nu_i{}^\frown d \trianglelefteq \nu_j$ for all $i, j \leq n$ and $d \leq 1$.

We say $\bar{\eta} \approx_\beta \bar{\nu}$ if they satisfy (i), (ii), (iii), and

(iv) $\eta_i{}^\frown\langle 1 \rangle = \eta_j$ if and only if $\nu_i{}^\frown\langle 1 \rangle = \nu_j$ for all $i, j \leq n$.

We say $\bar{\gamma} \approx_\gamma \bar{\nu}$ if they satisfy (i), (ii), (iii), (iv), and

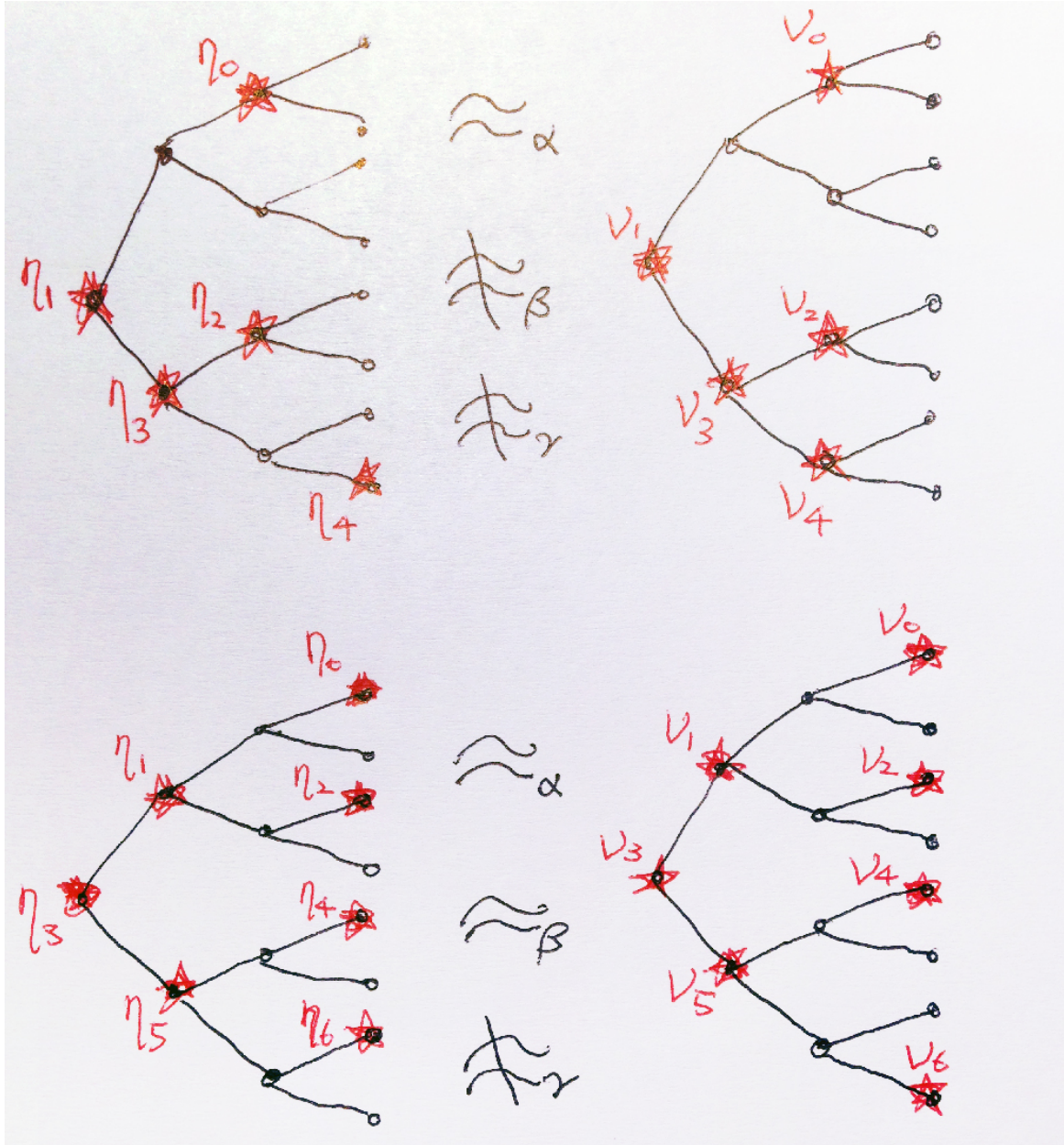
(v) $\eta_i{}^\frown\langle 0 \rangle = \eta_j$ if and only if $\nu_i{}^\frown\langle 0 \rangle = \nu_j$ for all $i, j \leq n$.

(vi) $\eta_i = \sigma{}^\frown\langle 0 \rangle$ for some $\sigma \in {}^{\omega>}2$ if and only if $\nu_i = \tau{}^\frown\langle 0 \rangle$ for some $\tau \in {}^{\omega>}2$, for all $i \leq n$.

(vii) $\eta_i = \sigma{}^\frown\langle 1 \rangle$ for some $\sigma \in {}^{\omega>}2$ if and only if $\nu_i = \tau{}^\frown\langle 1 \rangle$ for some $\tau \in {}^{\omega>}2$, for all $i \leq n$.

We say $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ is α-indiscernible (β, γ-indiscernible, resp.) if $\bar{\eta} \approx_\alpha \bar{\nu}$ ($\bar{\eta} \approx_\beta \bar{\nu}$, $\bar{\eta} \approx_\gamma \bar{\nu}$, resp.) implies $a_{\bar{\eta}} \equiv a_{\bar{\nu}}$.

Tree indiscernibility on SOP_1



Tree indiscernibility on SOP_1

Fact [H. Kim, B. Kim] (Modeling Property)

For any $\langle a_\eta \rangle_{\eta \in \omega^{>}2}$, there exists $\langle b_\eta \rangle_{\eta \in \omega^{>}2}$ such that

(i) *$\langle b_\eta \rangle_{\eta \in \omega^{>}2}$ is α-indiscernible,*

(ii) *for any finite set Δ of $\mathcal{L}$-formulas and $\wedge$-closed $\eta = \langle \eta_0, ..., \eta_n \rangle$, there exists $\bar{\nu} \approx_\alpha \bar{\eta}$ such that $\bar{b}_{\bar{\eta}} \equiv_\Delta \bar{a}_{\bar{\nu}}$.*

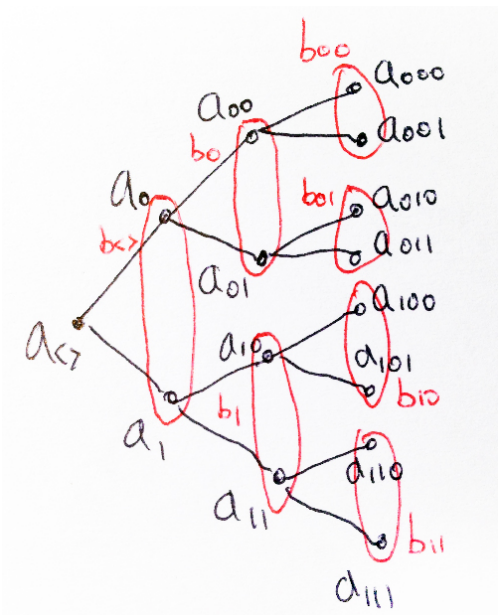
If a witness of SOP_1 has α-indiscernibility, then the witness infact witnesses SOP_2 ($\langle \eta, \eta \wedge \nu, \nu \rangle \approx_\alpha \langle \langle 0 \rangle, \langle \rangle, \langle 1 \rangle \rangle$ for any $\eta \perp \nu$). So if we want to handle witnesses SOP_1 without SOP_2 property, we need another indiscernibility which fits to SOP_1.

Tree indiscernibility on SOP_1

By using the modeling property, one can show that:

Lemma

Suppose $\varphi(x, y)$ *witnesses* SOP_1*. Then there is a* γ*-indiscernible tree* $\langle d_\eta \rangle_{\eta \in \omega^{>2}}$ *which witnesses* SOP_1 *with* φ*,*



sketch of proof

Let $\langle \varphi, \langle a_\eta \rangle \rangle$ be the witness of SOP_1.
For each node η of the given tree,
concatenate its direct siblings and name it b_η.
Apply the modeling property on $\langle b_\eta \rangle$
to obtain a new α-indiscernible tree $\langle c_\eta \rangle$.
Separate each node of $\langle c_\eta \rangle$,
then we have the γ-indiscernible SOP_1 witness.

Tree indiscernibility on SOP_1

And by appropriate restriction, we have

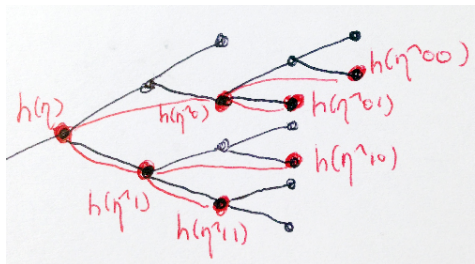
Theorem

Suppose $\varphi(x,y)$ *witnesses* SOP_1. *Then there is an* β*-indiscernible tree* $\langle e_\eta \rangle_{\eta \in {}^{\omega>}2}$ *which witnesses* SOP_1 *with* φ.

sketch of proof

By Lemma, there exists a γ-indiscernible tree $\langle d_\eta \rangle_{\eta \in {}^{\omega>}2}$ which witnesses SOP_1 with φ. Define a map $h : {}^{\omega>}2 \to {}^{\omega>}2$ by

$$h(\eta) := \begin{cases} \langle\rangle & \text{if} \quad \eta = \langle\rangle \\ h(\eta^-)^\frown\langle 01\rangle & \text{if} \quad t(\eta) = 0 \\ h(\eta^-)^\frown\langle 1\rangle & \text{if} \quad t(\eta) = 1. \end{cases}$$



Put $e_\eta := d_{h(\eta)}$ for each $\eta \in {}^{\omega>}2$. Then $\langle e_\eta \rangle_{\eta \in {}^{\omega>}2}$ is β-indiscernible and witnesses SOP_1 with φ (the map preserves every condition of $\approx_\beta$, and makes condition (v),(vi),(vii) vacuously true).

Construction of an antichain tree in SOP_1-NSOP_2

Definition

(i) A subset X of ${}^{\omega>}2$ is called an *antichain* if it is pairwisely incomparable (*i.e.* $\eta \perp \nu$ for all $\eta, \nu \in X$).

(ii) By an *antichain tree*, we mean a tuple $\langle \varphi(x, y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ which satisfies

$$\{\varphi(x, a_\eta) \mid \eta \in X\} \text{ is consistent} \Leftrightarrow X \text{ is an antichain}$$

for all $X \subseteq {}^{\omega>}2$.

Construction of an antichain tree in SOP_1-NSOP_2

Note that the notions of tree properties can be rewritten as follows.

(i) $\langle \varphi, \langle a_\eta \rangle \rangle$ is called an antichain tree if for all $X \subseteq {}^{\omega>}2$, $\{\varphi(x, a_\eta) | \eta \in X\}$ is consistent iff X is pairwisely incomparable,

(i) $\langle \varphi, \langle a_\eta \rangle \rangle$ is said to be a witness of SOP_2 if for all $X \subseteq {}^{\omega>}2$, $\{\varphi(x, a_\eta) | \eta \in X\}$ is consistent iff X is pairwisely comparable,

(iii) $\langle \varphi, \langle a_\eta \rangle \rangle$ is said to be a witness of SOP_1 if for all $X \subseteq {}^{\omega>}2$,

a. $\{\varphi(x, a_\eta) | \eta \in X\}$ is consistent if X is pairwisely comparable,

b. $\{\varphi(x, a_\eta) | \eta \in X\}$ is inconsistent if X contains $\eta^\frown\langle 1 \rangle$ and $\eta^\frown\langle 0 \rangle^\frown\nu$ for some $\eta, \nu \in {}^{\omega>}2$,

So if a witness of SOP_1 does not witness SOP_2, then the consistency of X becomes ambiguous when X is not pairwisely comparable. The main idea of construction of an antichain tree is restricting given SOP_1-tree into those ambiguous parts.

Construction of an antichain tree in SOP_1-NSOP_2

Theorem

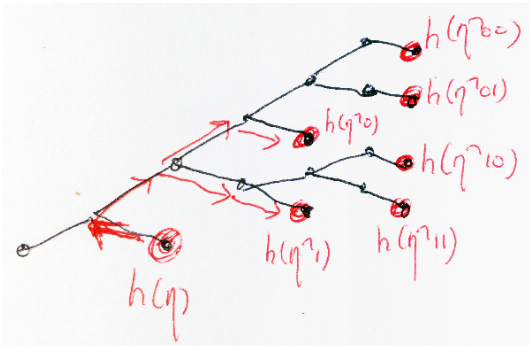
Suppose $\varphi(x, y)$ *witnesses* SOP_1 *and there is no* $n \in \omega$ *such that* $\bigwedge_{i=0}^{n} \varphi(x, y_i)$ *witnesses* SOP_2. *Then there exists* $\langle b_\eta \rangle_{\eta \in \omega^{>2}}$ *such that* $\langle \varphi(x, b_\eta) \rangle_{\eta \in \omega^{>2}} \rangle$ *forms an antichain tree.*

Construction of an antichain tree in SOP_1-NSOP_2

Sketch of proof

By the theorem above and compactness, there exists an β-indiscernible tree $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ which witnesses SOP_1 with φ. Define a map $h : {}^{\omega>}2 \to {}^{\omega_1>}2$ by

$$h(\eta) := \begin{cases} \langle 1 \rangle & \text{if} \quad \eta = \langle\rangle \\ h(\eta^-)^\frown \langle 001 \rangle & \text{if} \quad t(\eta) = 0 \\ h(\eta^-)^\frown \langle 011 \rangle & \text{if} \quad t(\eta) = 1. \end{cases}$$



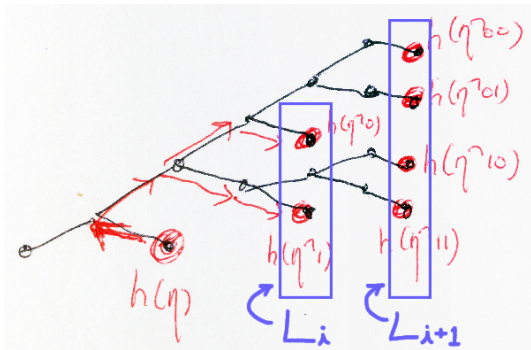
For each $i \in \omega$, we put $L_i = \{h(\eta) \mid l(\eta) = i\}$.
If $\{\varphi(x, a_\eta) | \eta \in L_i\}$ is inconsistent for some i, then one can construct SOP_2 by taking a suitable conjunction of $\varphi(x, a_\eta)$. So $\{\varphi(x, a_\eta) | \eta \in L_i\}$ is consistent for all i. Every h image of an antichain is β equivalence to a subset of L_i for some i, and every image of a comparable pair is of the form $\langle \eta^\frown \langle 1 \rangle, \eta^\frown \langle 0 \rangle^\frown \nu \rangle$. Thus $\langle \varphi(x, b_\eta) \rangle_{\eta \in \omega>2} \rangle$ forms an antichain tree, where $b_\eta = a_{h(\eta)}$.

Construction of an antichain tree in SOP_1-$NSOP_2$

Sketch of proof

By the theorem above and compactness, there exists an β-indiscernible tree $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ which witnesses SOP_1 with φ. Define a map $h : {}^{\omega>}2 \to {}^{\omega_1>}2$ by

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For each $i \in \omega$, we put $L_i = \{h(\eta) \mid l(\eta) = i\}$.
If $\{\varphi(x, a_\eta) | \eta \in L_i\}$ is inconsistent for some i, then one can construct SOP_2 by taking a suitable conjunction of $\varphi(x, a_\eta)$. So $\{\varphi(x, a_\eta) | \eta \in L_i\}$ is consistent for all i. Every h image of an antichain is β equivalence to a subset of L_i for some i, and every image of a comparable pair is of the form $\langle \eta^\frown \langle 1 \rangle, \eta^\frown \langle 0 \rangle^\frown \nu \rangle$. Thus $\langle \varphi(x, b_\eta) \rangle_{\eta \in {}^{\omega>}2} \rangle$ forms an antichain tree, where $b_\eta = a_{h(\eta)}$.

Construction of an antichain tree in SOP_1-$NSOP_2$

Definition

Let $L_0 = \{<_{lex}, <_{ini}, \cap\}$. For $\bar{\eta}, \bar{\nu} \in {}^{\omega>}2$, we say $\bar{\eta} \approx_{str} \bar{\nu}$ if $\bar{\eta}$ and $\bar{\nu}$ have the same L_0-atomic type (quantifier free type). We say $\langle a_\eta \rangle_{\eta \in \omega>2}$ is str-indiscernible if $\bar{\eta} \approx_{str} \bar{\nu}$ implies $a_{\bar{\eta}} \equiv a_{\bar{\nu}}$.

Fact [K. Takeuchi and A. Tsuboi]

Strong indiscerniblility has modeling property.

Remark

If $\varphi(x, y)$ forms an antichain tree, then there exists a str-indiscernible tree $\langle a_\eta \rangle_{\eta \in \omega>2}$ such that φ forms an antichain tree with $\langle a_\eta \rangle_{\eta \in \omega>2}$.

Corollary

If T is SOP_1 and $NSOP_2$, then T has an antichain tree $\langle \varphi(x, y), \langle a_\eta \rangle_{\eta \in \omega>2} \rangle$ such that $\langle a_\eta \rangle_{\eta \in \omega>2}$ is str-indiscernible.

Properties of antichain trees

Proposition

If $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in \omega^{>2}} \rangle$ *is an antichain tree, then* $\varphi(x,y)$ *witnesses* SOP_1.

Definition

We say T has TP_2 if there is a formula $\varphi(x,y)$ and tuples $\langle a_{i,j} \rangle_{i,j \in \omega}$ and $k \in \omega$ such that

(i) for each $f : \omega \to \omega$, the set $\{\varphi(x, a_{i,f(i)}) | i \in \omega\}$ is consistent,

(ii) for each $i \in \omega$, the set $\{\varphi(x, a_{i,j}) | j \in \omega\}$ is k-inconsistent.

Proposition

If $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in \omega^{>2}} \rangle$ *is an antichain tree, then* $\varphi(x,y)$ *witnesses* TP_2.

Properties of antichain trees

In general, every antichain tree contains SOP_1-tree and TP_2-array. By giving appropriate maps one can easily obtain SOP_1 and TP_2. But this method does not work for SOP_2.

Proposition

(i) Suppose $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ is an antichain tree. Then there is no $h : {}^{2>}2 \to {}^{\omega>}2$ such that $\langle \varphi(x,y), \langle b_\eta \rangle_{\eta \in {}^{2>}2} \rangle$ satisfies the conditions of SOP_2, where $b_\eta = a_{h(\eta)}$ for each $\eta \in {}^{2>}2$.

(ii) Suppose $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ witnesses SOP_2. Then there is no $h : {}^{2>}2 \to {}^{\omega>}2$ such that $\langle \varphi(x,y), \langle b_\eta \rangle_{\eta \in {}^{2>}2} \rangle$ forms an antichain tree with height 2, where $b_\eta = a_{h(\eta)}$ for each $\eta \in {}^{2>}2$.

Properties of antichain trees

Remark

(i) If there is an example of NSOP_2 theory having an antichain tree, then $\mathrm{SOP}_1 \neq \mathrm{SOP}_2$.

(ii) If the existence of an antichain tree implies SOP_2, then $\mathrm{SOP}_1 = \mathrm{SOP}_2$.

(iii) If there is a general way to construct an antichain tree in a $\mathrm{SOP}_n \cap \mathrm{NSOP}_{n+1}$ theory, then SOP_n hierarchy collapse in NTP_2.)

What we have constructed so far

There is a theory T such that

(i) T has an antichain tree $\langle \varphi, \langle a_\eta \rangle \rangle$ where φ is an atomic formula,

(ii) T has no quantifier free formula which witnesses SOP_2.

Since φ is quantifier free, $\bigwedge_n \varphi$ does not witness SOP_2 for any $n \in \omega$.

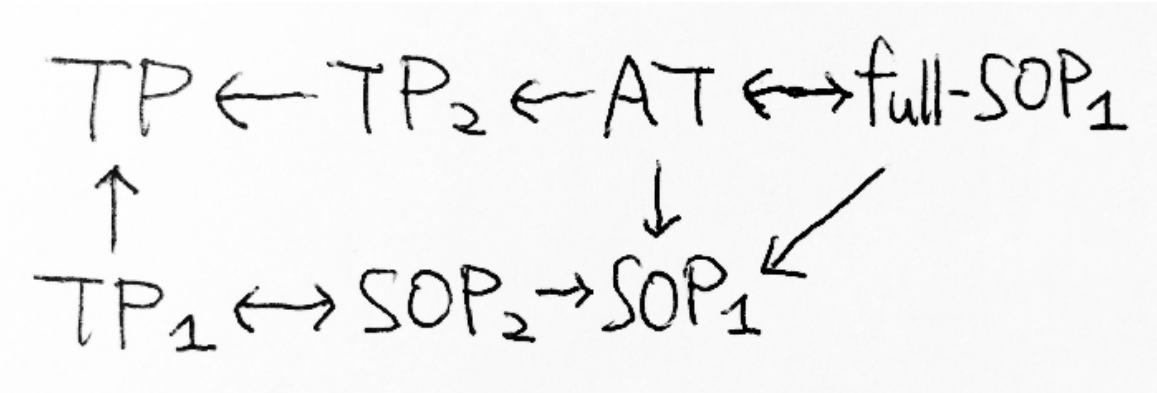
Properties of antichain trees

Definition

$\langle \varphi, \langle a_\eta \rangle \rangle$ is said to be a witness of full-SOP$_1$ if for all $X \subseteq {}^{\omega>}2$, $\{\varphi(x, a_\eta) | \eta \in X\}$ is consistent iff X does not contain $\eta^\frown\langle 1 \rangle$ and $\eta^\frown\langle 0 \rangle^\frown\nu$ for some $\eta, \nu \in {}^{\omega>}2$.

Proposition

If a theory T has an antichain tree iff T has a witness of full-SOP$_1$.



References

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- K. Takeuchi and A. Tsuboi. On the existence of indiscernible trees. Annals of Pure and Applied Logic 163 (2012), 1891-1902.
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