# Antichain tree property and 1-strictly strong order property

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July 3, 2020

# Outline

Tree indiscernibility on $\mathrm{SOP}_1$

Construction of an antichain tree in $\mathrm{SOP}_1$-$\mathrm{NSOP}_2$

Properties of antichain trees

1-strictly strong order property

# Tree indiscernibility on $\mathrm{SOP}_1$

## Definition

We say a theory $T$ has $\mathrm{SOP}_1$ if there is a formula $\varphi(x, y)$ and tuples $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ such that

(i) for each $\eta \in {}^{\omega}2$, $\{\varphi(x, a_{\eta \restriction n}) | n \in \omega\}$ is consistent,

(ii) for each $\eta, \nu \in {}^{\omega>}2$, $\{\varphi(x, a_{\eta^\frown \langle 1 \rangle}), \varphi(x, a_{\eta^\frown \langle 0 \rangle^\frown \nu})\}$ is inconsistent.

We say $T$ has $\mathrm{SOP}_2$ if there is a formula $\varphi(x, y)$ and tuples $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ such that

(i) for each $\eta \in {}^{\omega}2$, the set $\{\varphi(x, a_{\eta \restriction n}) | n \in \omega\}$ is consistent,

(ii) for each $\eta, \nu \in {}^{\omega>}2$ with $\eta \not\trianglelefteq \nu$ and $\nu \not\trianglelefteq \eta$, the set $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent.

## Fact

*If $T$ has $SOP_2$, then $T$ has $SOP_1$.*

But we do not know whether the converse holds or not.

# Tree indiscernibility on $\mathrm{SOP}_1$

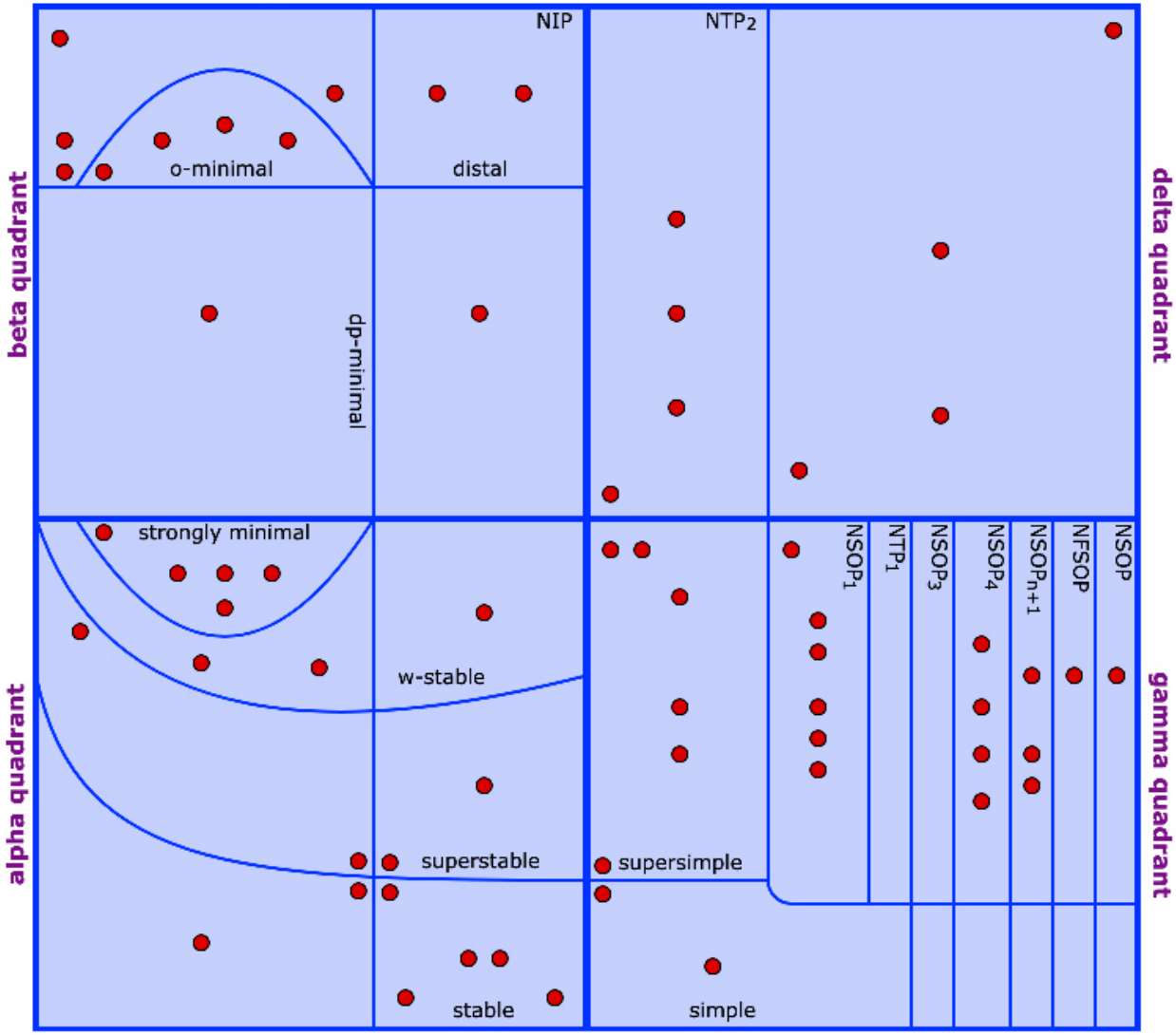
## Example

Let $\mathcal{L} = \{<\}$ and $T = Th(\mathbb{R})$ (*i.e.* the set of all true $\mathcal{L}$-sentence in $\mathbb{R}$), $\varphi(x, y_0, y_1) := y_0 < x < y_1$. For each $\eta \in {}^{\omega>}2$, choose $r_{\eta,0}, r_{\eta,1} \in \mathbb{R}$ by

$$r_{\eta,0} := \Sigma_{i \in \mathrm{dom}(\eta)} \eta(i) 2^{-(i+1)}$$
$$r_{\eta,1} := r_{\eta,0} + 2^{-l(\eta)}$$

where $l(\eta)$ is the length of $\eta$. Then $\langle \varphi(x, y_0, y_1), \langle r_{\eta,0}, r_{\eta,1} \rangle \rangle_{\eta \in {}^{\omega>}2}$ witnesses $\mathrm{SOP}_2$ modulo $T$.

# forking and dividing



# Tree indiscernibility on $\mathrm{SOP}_1$

## Definition

For $\bar{\eta} = \langle \eta_0, ..., \eta_n \rangle$, $\bar{\nu} = \langle \nu_0, ..., \nu_n \rangle$ ($\eta_i, \nu_i \in {}^{\omega>}2$), we say $\bar{\eta} \approx_\alpha \bar{\nu}$ if they satisfies

(i) $\bar{\eta}$ and $\bar{\nu}$ are $\wedge$-closed,

(ii) $\eta_i \trianglelefteq \eta_j$ if and only if $\nu_i \trianglelefteq \nu_j$ for all $i, j \leq n$,

(iii) $\eta_i{}^\frown d \trianglelefteq \eta_j$ if and only if $\nu_i{}^\frown d \trianglelefteq \nu_j$ for all $i, j \leq n$ and $d \leq 1$.

We say $\bar{\eta} \approx_\beta \bar{\nu}$ if they satisfy (i), (ii), (iii), and

(iv) $\eta_i{}^\frown\langle 1 \rangle = \eta_j$ if and only if $\nu_i{}^\frown\langle 1 \rangle = \nu_j$ for all $i, j \leq n$.

We say $\bar{\eta} \approx_\gamma \bar{\nu}$ if they satisfy (i), (ii), (iii), (iv), and

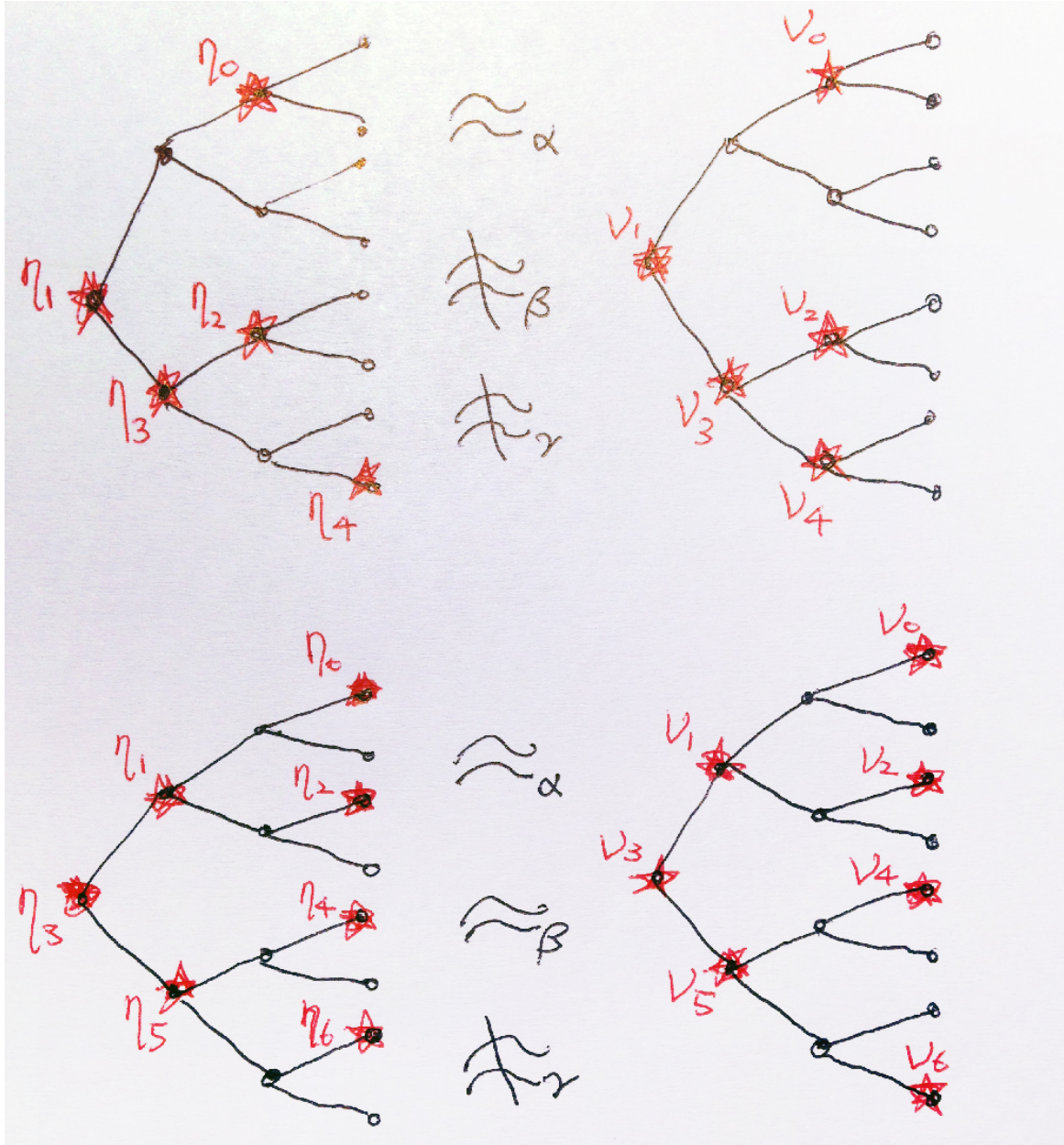
(v) $\eta_i{}^\frown\langle 0 \rangle = \eta_j$ if and only if $\nu_i{}^\frown\langle 0 \rangle = \nu_j$ for all $i, j \leq n$.

(vi) $\eta_i = \sigma{}^\frown\langle 0 \rangle$ for some $\sigma \in {}^{\omega>}2$ if and only if $\nu_i = \tau{}^\frown\langle 0 \rangle$ for some $\tau \in {}^{\omega>}2$, for all $i \leq n$.

(vii) $\eta_i = \sigma{}^\frown\langle 1 \rangle$ for some $\sigma \in {}^{\omega>}2$ if and only if $\nu_i = \tau{}^\frown\langle 1 \rangle$ for some $\tau \in {}^{\omega>}2$, for all $i \leq n$.

We say $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ is $\alpha$-indiscernible ($\beta$, $\gamma$-indiscernible, resp.) if $\bar{\eta} \approx_\alpha \bar{\nu}$ ($\bar{\eta} \approx_\beta \bar{\nu}$, $\bar{\eta} \approx_\gamma \bar{\nu}$, resp.) implies $a_{\bar{\eta}} \equiv a_{\bar{\nu}}$.

# Tree indiscernibility on $\mathrm{SOP}_1$



# Tree indiscernibility on $\mathrm{SOP}_1$

## Fact [H. Kim, B. Kim] (Modeling Property)

*For any $\langle a_\eta \rangle_{\eta \in \omega^{>}2}$, there exists $\langle b_\eta \rangle_{\eta \in \omega^{>}2}$ such that*

(i) *$\langle b_\eta \rangle_{\eta \in \omega^{>}2}$ is $\alpha$-indiscernible,*

(ii) *for any finite set $\Delta$ of $\mathcal{L}$-formulas and $\wedge$-closed $\eta = \langle \eta_0, ..., \eta_n \rangle$, there exists $\bar{\nu} \approx_\alpha \bar{\eta}$ such that $\bar{b}_{\bar{\eta}} \equiv_\Delta \bar{a}_{\bar{\nu}}$.*

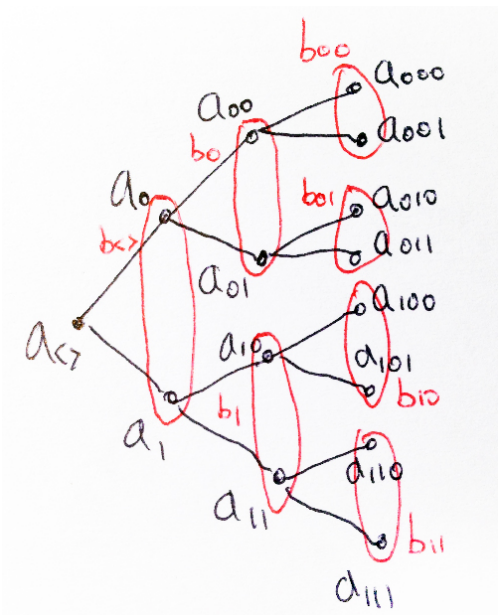
If a witness of $\mathrm{SOP}_1$ has $\alpha$-indiscernibility, then the witness infact witnesses $\mathrm{SOP}_2$ ($\langle \eta, \eta \wedge \nu, \nu \rangle \approx_\alpha \langle \langle 0 \rangle, \langle \rangle, \langle 1 \rangle \rangle$ for any $\eta \perp \nu$). So if we want to handle witnesses $\mathrm{SOP}_1$ without $\mathrm{SOP}_2$ property, we need another indiscernibility which fits to $\mathrm{SOP}_1$.

# Tree indiscernibility on $\mathrm{SOP}_1$

By using the modeling property, one can show that:

## Lemma

*Suppose* $\varphi(x, y)$ *witnesses* $\mathrm{SOP}_1$*. Then there is a* $\gamma$*-indiscernible tree* $\langle d_\eta \rangle_{\eta \in \omega^{>2}}$ *which witnesses* $\mathrm{SOP}_1$ *with* $\varphi$*,*




## sketch of proof

Let $\langle \varphi, \langle a_\eta \rangle \rangle$ be the witness of $\mathrm{SOP}_1$.
For each node $\eta$ of the given tree,
concatenate its direct siblings and name it $b_\eta$.
Apply the modeling property on $\langle b_\eta \rangle$
to obtain a new $\alpha$-indiscernible tree $\langle c_\eta \rangle$.
Separate each node of $\langle c_\eta \rangle$,
then we have the $\gamma$-indiscernible $\mathrm{SOP}_1$ witness.

# Tree indiscernibility on $\mathrm{SOP}_1$

And by appropriate restriction, we have

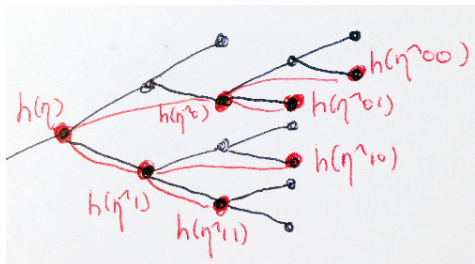
## Theorem

*Suppose* $\varphi(x,y)$ *witnesses* $\mathrm{SOP}_1$. *Then there is an* $\beta$*-indiscernible tree* $\langle e_\eta \rangle_{\eta \in {}^{\omega>}2}$ *which witnesses* $\mathrm{SOP}_1$ *with* $\varphi$.

## sketch of proof

By Lemma, there exists a $\gamma$-indiscernible tree $\langle d_\eta \rangle_{\eta \in {}^{\omega>}2}$ which witnesses $\mathrm{SOP}_1$ with $\varphi$. Define a map $h : {}^{\omega>}2 \to {}^{\omega>}2$ by

$$h(\eta) := \begin{cases} \langle\rangle & \text{if} \quad \eta = \langle\rangle \\ h(\eta^-)^\frown\langle 01\rangle & \text{if} \quad t(\eta) = 0 \\ h(\eta^-)^\frown\langle 1\rangle & \text{if} \quad t(\eta) = 1. \end{cases}$$




Put $e_\eta := d_{h(\eta)}$ for each $\eta \in {}^{\omega>}2$. Then $\langle e_\eta \rangle_{\eta \in {}^{\omega>}2}$ is $\beta$-indiscernible and witnesses $\mathrm{SOP}_1$ with $\varphi$ (the map preserves every condition of $\approx_\beta$, and makes condition (v),(vi),(vii) vacuously true).

# Construction of an antichain tree in $\mathrm{SOP}_1$-$\mathrm{NSOP}_2$

## Definition

(i) A subset $X$ of ${}^{\omega>}2$ is called an *antichain* if it is pairwisely incomparable (*i.e.* $\eta \perp \nu$ for all $\eta, \nu \in X$).

(ii) By an *antichain tree*, we mean a tuple $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ which satisfies

$$\{\varphi(x, a_\eta) \mid \eta \in X\} \text{ is consistent} \Leftrightarrow X \text{ is an antichain}$$

for all $X \subseteq {}^{\omega>}2$.

(iii) We say a theory $T$ has *antichain tree property (ATP)* if there is a formula $\varphi$ and $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ such that $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ forms an antichain tree modulo $T$.

# Construction of an antichain tree in $\mathrm{SOP}_1$-$\mathrm{NSOP}_2$

Note that the notions of tree properties can be rewritten as follows.

(i) $\langle \varphi, \langle a_\eta \rangle \rangle$ is called an antichain tree if for all $X \subseteq {}^{\omega>}2$, $\{\varphi(x, a_\eta) | \eta \in X\}$ is consistent iff $X$ is pairwisely incomparable,

(i) $\langle \varphi, \langle a_\eta \rangle \rangle$ is said to be a witness of $\mathrm{SOP}_2$ if for all $X \subseteq {}^{\omega>}2$, $\{\varphi(x, a_\eta) | \eta \in X\}$ is consistent iff $X$ is pairwisely comparable,

(iii) $\langle \varphi, \langle a_\eta \rangle \rangle$ is said to be a witness of $\mathrm{SOP}_1$ if for all $X \subseteq {}^{\omega>}2$,

a. $\{\varphi(x, a_\eta) | \eta \in X\}$ is consistent if $X$ is pairwisely comparable,

b. $\{\varphi(x, a_\eta) | \eta \in X\}$ is inconsistent if $X$ contains $\eta^\frown\langle 1 \rangle$ and $\eta^\frown\langle 0 \rangle^\frown\nu$ for some $\eta, \nu \in {}^{\omega>}2$,

So if a witness of $\mathrm{SOP}_1$ does not witness $\mathrm{SOP}_2$, then the consistency of $X$ becomes ambiguous when $X$ is not pairwisely comparable. The main idea of construction of an antichain tree is restricting given $\mathrm{SOP}_1$-tree into those ambiguous parts.

# Construction of an antichain tree in $\mathrm{SOP}_1$-$\mathrm{NSOP}_2$

## Theorem

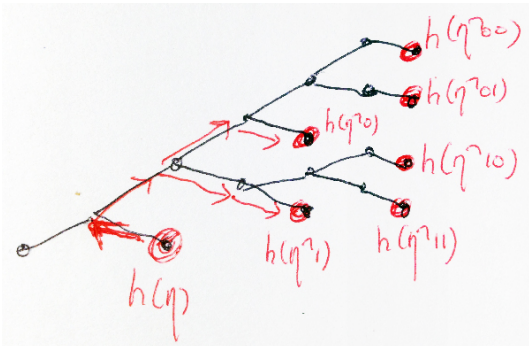
*Suppose* $\varphi(x, y)$ *witnesses* $\mathrm{SOP}_1$ *and there is no* $n \in \omega$ *such that* $\bigwedge_{i=0}^{n} \varphi(x, y_i)$ *witnesses* $\mathrm{SOP}_2$. *Then there exists* $\langle b_\eta \rangle_{\eta \in \omega^{>2}}$ *such that* $\langle \varphi(x, b_\eta) \rangle_{\eta \in \omega^{>2}} \rangle$ *forms an antichain tree.*

# Construction of an antichain tree in $\mathrm{SOP}_1$-$\mathrm{NSOP}_2$

## Sketch of proof

By the theorem above and compactness, there exists an $\beta$-indiscernible tree $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ which witnesses $\mathrm{SOP}_1$ with $\varphi$. Define a map $h : {}^{\omega>}2 \to {}^{\omega_1>}2$ by

$$h(\eta) := \begin{cases} \langle 1 \rangle & \text{if} \quad \eta = \langle\rangle \\ h(\eta^-)^\frown \langle 001 \rangle & \text{if} \quad t(\eta) = 0 \\ h(\eta^-)^\frown \langle 011 \rangle & \text{if} \quad t(\eta) = 1. \end{cases}$$



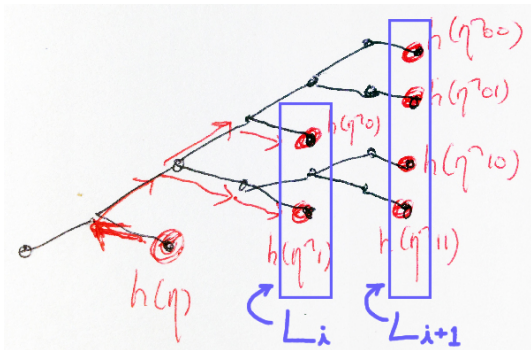

For each $i \in \omega$, we put $L_i = \{h(\eta) \mid l(\eta) = i\}$.
If $\{\varphi(x, a_\eta) | \eta \in L_i\}$ is inconsistent for some $i$, then one can construct $\mathrm{SOP}_2$ by taking a suitable conjunction of $\varphi(x, a_\eta)$. So $\{\varphi(x, a_\eta) | \eta \in L_i\}$ is consistent for all $i$. Every $h$ image of an antichain is $\beta$ equivalence to a subset of $L_i$ for some $i$, and every image of a comparable pair is of the form $\langle \eta^\frown \langle 1 \rangle, \eta^\frown \langle 0 \rangle^\frown \nu \rangle$. Thus $\langle \varphi(x, b_\eta) \rangle_{\eta \in \omega>2} \rangle$ forms an antichain tree, where $b_\eta = a_{h(\eta)}$.

# Construction of an antichain tree in $SOP_1$-$NSOP_2$

## Sketch of proof

By the theorem above and compactness, there exists an $\beta$-indiscernible tree $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ which witnesses $SOP_1$ with $\varphi$. Define a map $h : {}^{\omega>}2 \to {}^{\omega_1>}2$ by

$$h(\eta) := \begin{cases} \langle 1 \rangle & \text{if} \quad \eta = \langle\rangle \\ h(\eta^-)^\frown \langle 001 \rangle & \text{if} \quad t(\eta) = 0 \\ h(\eta^-)^\frown \langle 011 \rangle & \text{if} \quad t(\eta) = 1. \end{cases}$$




For each $i \in \omega$, we put $L_i = \{h(\eta) \mid l(\eta) = i\}$.
If $\{\varphi(x, a_\eta) | \eta \in L_i\}$ is inconsistent for some $i$, then one can construct $SOP_2$ by taking a suitable conjunction of $\varphi(x, a_\eta)$. So $\{\varphi(x, a_\eta) | \eta \in L_i\}$ is consistent for all $i$. Every $h$ image of an antichain is $\beta$ equivalence to a subset of $L_i$ for some $i$, and every image of a comparable pair is of the form $\langle \eta^\frown \langle 1 \rangle, \eta^\frown \langle 0 \rangle^\frown \nu \rangle$. Thus $\langle \varphi(x, b_\eta) \rangle_{\eta \in {}^{\omega>}2} \rangle$ forms an antichain tree, where $b_\eta = a_{h(\eta)}$.

# Construction of an antichain tree in $SOP_1$-$NSOP_2$

## Definition

Let $L_0 = \{<_{lex}, <_{ini}, \cap\}$. For $\bar{\eta}, \bar{\nu} \in {}^{\omega>}2$, we say $\bar{\eta} \approx_{str} \bar{\nu}$ if $\bar{\eta}$ and $\bar{\nu}$ have the same $L_0$-atomic type (quantifier free type). We say $\langle a_\eta \rangle_{\eta \in \omega>2}$ is str-indiscernible if $\bar{\eta} \approx_{str} \bar{\nu}$ implies $a_{\bar{\eta}} \equiv a_{\bar{\nu}}$.

## Fact [K. Takeuchi and A. Tsuboi]

Strong indiscerniblility has modeling property.

## Remark

If $\varphi(x, y)$ forms an antichain tree, then there exists a str-indiscernible tree $\langle a_\eta \rangle_{\eta \in \omega>2}$ such that $\varphi$ forms an antichain tree with $\langle a_\eta \rangle_{\eta \in \omega>2}$.

## Corollary

*If $T$ is $SOP_1$ and $NSOP_2$, then $T$ has an antichain tree $\langle \varphi(x, y), \langle a_\eta \rangle_{\eta \in \omega>2} \rangle$ such that $\langle a_\eta \rangle_{\eta \in \omega>2}$ is str-indiscernible.*

# Properties of antichain trees

## Proposition

*If* $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in \omega^{>2}} \rangle$ *is an antichain tree, then* $\varphi(x,y)$ *witnesses* $\mathrm{SOP}_1$.

## Definition

We say $T$ has $\mathrm{TP}_2$ if there is a formula $\varphi(x,y)$ and tuples $\langle a_{i,j} \rangle_{i,j \in \omega}$ and $k \in \omega$ such that

(i) for each $f : \omega \to \omega$, the set $\{\varphi(x, a_{i,f(i)}) | i \in \omega\}$ is consistent,

(ii) for each $i \in \omega$, the set $\{\varphi(x, a_{i,j}) | j \in \omega\}$ is $k$-inconsistent.

## Proposition

*If* $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in \omega^{>2}} \rangle$ *is an antichain tree, then* $\varphi(x,y)$ *witnesses* $\mathrm{TP}_2$.

# Properties of antichain trees

In general, every antichain tree contains $\mathrm{SOP}_1$-tree and $\mathrm{TP}_2$-array. By giving appropriate maps one can easily obtain $\mathrm{SOP}_1$ and $\mathrm{TP}_2$. But this method does not work for $\mathrm{SOP}_2$.

## Proposition

(i) Suppose $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ is an antichain tree. Then there is no $h : {}^{2>}2 \to {}^{\omega>}2$ such that $\langle \varphi(x,y), \langle b_\eta \rangle_{\eta \in {}^{2>}2} \rangle$ satisfies the conditions of $\mathrm{SOP}_2$, where $b_\eta = a_{h(\eta)}$ for each $\eta \in {}^{2>}2$.

(ii) Suppose $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ witnesses $\mathrm{SOP}_2$. Then there is no $h : {}^{2>}2 \to {}^{\omega>}2$ such that $\langle \varphi(x,y), \langle b_\eta \rangle_{\eta \in {}^{2>}2} \rangle$ forms an antichain tree with height 2, where $b_\eta = a_{h(\eta)}$ for each $\eta \in {}^{2>}2$.

# Properties of antichain trees

## Remark

(i) If there is an example of $\mathrm{NSOP}_2$ theory having an antichain tree, then $\mathrm{SOP}_1 \neq \mathrm{SOP}_2$.

(ii) If the existence of an antichain tree implies $\mathrm{SOP}_2$, then $\mathrm{SOP}_1 = \mathrm{SOP}_2$.

(iii) If there is a general way to construct an antichain tree in a $\mathrm{SOP}_n \cap \mathrm{NSOP}_{n+1}$ theory, then $\mathrm{SOP}_n$ hierarchy collapse in $\mathrm{NTP}_2$.)

## Question

Is there a $\mathrm{NSOP}_2$ theory which has an antichain tree?

There is a theory $T$ such that

(i) $T$ has an antichain tree $\langle \varphi, \langle a_\eta \rangle \rangle$ where $\varphi$ is an atomic formula,

(ii) $T$ has no quantifier free formula which witnesses $\mathrm{SOP}_2$.

Since $\varphi$ is quantifier free, $\bigwedge_n \varphi$ does not witness $\mathrm{SOP}_2$ for any $n \in \omega$.

# 1-strictly strong order property

## Definition

(i) We say a tuple $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ witnesses *1-strictly strong order property (SSOP$_1$)* if for all $X \subseteq {}^{\omega>}2$, $\{\varphi(x,a_\eta) : \eta \in X\}$ is consistent if and only if $X$ does not contain $\eta^\frown\langle 1\rangle$ and $\eta^\frown\langle 0\rangle^\frown\nu$ for some $\eta, \nu \in {}^{\omega>}2$.

(ii) We say a theory has SSOP$_1$ if it has an SSOP$_1$ formula. Otherwise, we say $T$ is NSSOP$_1$.

## Remark/Proposition

(i) If $T$ has SSOP$_1$, then it has SOP$_1$ by definition. But the converse does not hold in general.

(ii) Suppose $\varphi(x,y)$ witnesses SSOP$_1$. Then there is a $\beta$-indiscernible tree which witnesses SSOP$_1$ with $\varphi$.

(iii) If a theory $T$ has an antichain tree iff $T$ has a witness of SSOP$_1$.

# 1-strictly strong order property

Let $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in {}^{\omega>}2} \rangle$ is a witness of $\mathrm{SOP}_1/\mathrm{SOP}_2/\mathrm{SSOP}_1$ and $X \subseteq {}^{\omega>}2$. Then the consistency of $\{\varphi(x,a_\eta) : \eta \in X\}$ follows the table below:

| Case | $\mathrm{SOP}_1$ | $\mathrm{SOP}_2$ | $\mathrm{SSOP}_1$ |
|---|---|---|---|
| (a) | consistent | consistent | consistent |
| (b) | inconsistent | inconsistent | inconsistent |
| (c) | ? | inconsistent | consistent |

where

(a) $X$ is linearly ordered by $\trianglelefteq$

(b) $\eta^\frown\langle 1\rangle, \eta^\frown\langle 0\rangle^\frown\nu \in X$ for some $\eta, \nu \in {}^{\omega>}2$

(c) neither (a) nor (b).

# 1-strictly strong order property

## Fact

*$T$ has $SOP_1$ if and only if there exist $\varphi(x,y)$ and $\langle a_{i,j}\rangle_{i\in\omega}^{j<2}$ such that:*

(i) $a_{i,0} \equiv_{a_{<i,0},a_{<i,1}} a_{i,1}$ *for all* $i \in \omega$

(ii) $\{\varphi(x,a_{i,0}) : i \in \omega\}$ *is consistent*

(iii) $\{\varphi(x,a_{i,1}) : i \in \omega\}$ *is 2-inconsistent.*

## Question

Is there an equivalent condition for $\mathrm{SSOP}_1$ in the same way as in the case of $\mathrm{SOP}_1$ above?

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