Some Remarks on Tree Properties

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Outline

Modeling properties on ill-founded trees and its application

Construction of an ATP structure which has no quantifier free SOP_2 witness

Modeling properties on ill-founded trees and its application

Definition

Let α be an ordinal. $\mathcal{T}_\alpha$ is the set of functions which satisfies:

(i) its domain is an end-segment of α of the form $[\beta, \alpha)$ for β is equal to 0 or a successor ordinal. If α is successor, we allow $\beta = \alpha$.

(ii) its range is a subset of ω,

(iii) [finite support] there are only finitely many elements in its domain whose image is nonzero.

$\mathcal{T}_\omega$ is a tree which has no least element, whose paths are countable.

Modeling properties on ill-founded trees and its application

Definition

Let I is an $\mathcal{L}'$-structure, where $\mathcal{L}'$ is a language.

(i) We say $(a_i)_{i\in I}$ is I-indexed indiscernible if $qftp_{\mathcal{L}'}(i_0,...,i_n) = qftp_{\mathcal{L}'}(j_0,...,j_n)$ implies $a_{i_0}...a_{i_n} \equiv a_{j_0}...a_{j_n}$.

(ii) We say $(b_i)_{i\in I}$ is locally based on $(a_i)_{i\in I}$ if for each $i_0,...,i_n \in I$ and finite set of formula Δ from $\mathcal{L}$, there exists $j_0,...,j_n \in I$ such that $qftp_{\mathcal{L}'}(i_0,...,i_n) = qftp_{\mathcal{L}'}(j_0,...,j_n)$ and $a_{i_0}...a_{i_n} \equiv_\Delta b_{j_0}...b_{j_n}$.

(iii) We say I-indexed indiscernibles have the modeling property if for each I-indexed set $(a_i)_{i\in I}$, there exists I-indexed indiscernible $(b_i)_{i\in I}$ which is locally based on $(a_i)_{i\in I}$.

Remark

$\mathcal{T}_\alpha$ appears in [I.Kaplan, N.Ramsey] in order to find generic sequences in NSOP_1 structures. If $a \mathop{\smile\!\!\!\!\!\!|}^{K} b$ over a model, then we can construct a Morley tree which is indexed by $\mathcal{T}_\alpha$. The leftmost path of the Morley tree which is called a tree Morley sequence, witnesses of Kim-dividing and this observation allows us to conclude that NSOP_1 theories can be characterized in terms of Kim-independence. In [J.Dobrowolsky, B. Kim, and N. Ramsey], this result is extended to the case of $a \mathop{\smile\!\!\!\!\!\!|}^{K} b$ over an arbitrary set, under the existence axiom, using $\mathcal{T}_\alpha$.

Modeling properties on ill-founded trees and its application

Let $\mathcal{L}_{str} = \{\trianglelefteq, \wedge, <_{lex}\}$, $\mathcal{L}_{ss} = \{\trianglelefteq, \wedge, <_{lex}, <_{len}\}$, $\mathcal{L}_{s,\alpha} = \{\trianglelefteq, \wedge, <_{lex}, P_\beta\}_{\beta<\alpha}$ where α is an ordinal. Then $\omega^{<\omega}$ can be regarded as $\mathcal{L}_{str}$, $\mathcal{L}_{ss}$, $\mathcal{L}_{s,\alpha}$-structures by interpreting $\trianglelefteq, \wedge, <_{lex}, <_{len}$, and P_β as

- $\eta \trianglelefteq \nu$ iff $\eta \subseteq \nu$,
- $\eta \wedge \nu = \eta_{\restriction m}$,
- $\eta <_{lex} \nu$ iff $\eta \trianglelefteq \nu$ or $\nu \not\trianglelefteq \eta$ and $\eta(m) < \nu(m)$,
- $\eta <_{len} \nu$ iff $\mathrm{dom}(\eta) < \mathrm{dom}(\nu)$,
- and $P_\beta(\eta)$ iff $\mathrm{dom}(\eta) = \beta$, respectively,

where $m = \min\{n \in \omega : \eta(n) \neq \nu(n)\}$. Similarly we can regard $\mathcal{T}_\alpha$ as $\mathcal{L}_{str}$, $\mathcal{L}_{ss}$, $\mathcal{L}_{s,\alpha}$-structures.

Theorem [K. Takeuchi, A. Tsuboi]

$\omega^{<\omega}$ has $\mathcal{L}_{s,\omega}$, $\mathcal{L}_{ss}$, $\mathcal{L}_{str}$ modeling properties

Modeling properties on ill-founded trees and its application

Remark

By the same argument in the proof of above theorem, we can show that $\mathcal{T}_\alpha$ also has $\mathcal{L}_{s,\omega}$, $\mathcal{L}_{ss}$, $\mathcal{L}_{str}$ modeling properties. $\mathcal{L}_{ss}$, $\mathcal{L}_{str}$ modeling property of $\mathcal{T}_\alpha$ also can be shown by using the following theorem.

Theorem [L. Scow]

If I is a locally finite structure in a finite language $\mathcal{L}'$, then I-indexed indiscernibles have the modeling property if and only if age(I) is a Ramsey class.

By "I is locally finite" we mean every finitely generated substructure is finite. Clearly age($\omega^{<\omega}$) = age($\mathcal{T}_\omega$). Thus we can apply these theorems to obtain the $\mathcal{L}_{ss}$, $\mathcal{L}_{str}$ modeling property of $\mathcal{T}_\omega$.

Modeling properties on ill-founded trees and its application

Mutually tree-indiscernible sequences of trees

A sequence of trees $\langle A^i \rangle_{i \in \omega} = \langle \langle a^i_\eta \rangle_{\eta \in \mathcal{T}_\alpha} \rangle_{i \in \omega}$ is said to be mutually s-indiscernible over a set B if A^i is s-indiscernible over $BA^{\neq i}$ for each $i \in \omega$. Similarly we define mutually ss-indiscernible sequences of trees and mutually str-indiscernible sequences of trees.

Remark

Let $\alpha \leq \omega$ and suppose $A = \langle a_\eta \rangle_{\eta \in \mathcal{T}_\alpha}$ is an s-indiscernible tree over a model M. Then there exists a mutually s-indiscernible sequence of trees $\langle A^i \rangle_{i \in \omega} = \langle \langle a^i_\eta \rangle_{\eta \in \mathcal{T}_\alpha} \rangle_{i \in \omega}$ over M and a global M-invariant type q which satisfy

- $A^0 = A$,
- $A^i \models q_{\lceil MA^{<i}}$ for all $i \in \omega$.

Similarly there exists a mutually ss-(str-)indiscernible sequence of trees $\langle A^i \rangle_{i \in \omega}$ and a global M-invariant type q such that

- $A^0 = A$,
- $A^i \models q_{\lceil MA^{<i}}$ for all $i \in \omega$.

Modeling properties on ill-founded trees and its application

Remark

Let $\alpha \leq \omega$ and suppose $A = \langle a_\eta \rangle_{\eta \in \mathcal{T}_\alpha}$ is an s-indiscernible tree over a model M. Then there exists a mutually s-indiscernible sequence of trees $\langle A^i \rangle_{i \in \omega} = \langle \langle a^i_\eta \rangle_{\eta \in \mathcal{T}_\alpha} \rangle_{i \in \omega}$ over M and a global M-invariant type q which satisfy

- $A^0 = A$,
- $A^i \models q_{\lceil MA^{<i}}$ for all $i \in \omega$.

sketch of proof. Let $A^0 = A$ and q_0 be a global M-invariant type containing $\mathrm{tp}(A^0/M)$. Let N be a $|MA^0|^+$-saturated model containing MA^0 and $A^{0.5} = \langle a^1_\eta \rangle_{\eta \in \mathcal{T}_\alpha}$ be a realization of $q_0 \lceil N$. By the s-modeling property, we can find an s-indiscernible tree A^1 over N, which is s-based on $A^{0.5}$ over N. Then $\mathrm{tp}(A^1/N)$ is extending $\mathrm{tp}(A^0/M)$, M-invariant, A^1 is s-indiscernible over MA^0. We use defining schema, we can find M-invariant global extension of $\mathrm{tp}(A^1/N)$, say q. For each $i > 1$, let A^i be a realization of $q_{\lceil NA^{<i}}$. By M-invariance of q and the choice of A^i, the sequence $\langle A^i \rangle_{i \in \omega}$ is a mutually s-indiscernible sequence of trees.

Modeling properties on ill-founded trees and its application

Morley trees and spread-outness

- We say $\langle a_\eta \rangle_{\eta \in \mathcal{T}_\alpha}$ is spread out over B if there exists a global B-invariant type q extending $\mathrm{tp}(\langle a_{\trianglerighteq \eta^\frown \langle 0 \rangle} \rangle / B)$ for each $\eta \in \mathcal{T}_\alpha$, which satisfies $\langle a_{\trianglerighteq \eta^\frown \langle i \rangle} \rangle \models q_{\lceil B \langle a_{\trianglerighteq \eta^\frown \langle j \rangle} \rangle_{j<i}}$ for each $i \in \omega$.
- We say $\langle a_\eta \rangle_{\eta \in \mathcal{T}_\alpha}$ is a Morley tree if it is spread out and ss-indiscernible.

Remark

Let $\alpha \leq \omega$ and $\langle a_\eta \rangle_{\eta \in \mathcal{T}_\alpha}$ be a tree which is spread out over A. Suppose $\langle a_{\trianglerighteq \eta^\frown \langle i \rangle} \rangle_{i \in \omega}$ is mutually s-indiscernible over A and order indiscernible over Aa_η for each $\eta \in \mathcal{T}_\alpha$. If $\langle b_\eta \rangle_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over A and s-based on $\langle a_\eta \rangle_{\eta \in \mathcal{T}_\alpha}$ over A, then $\langle b_\eta \rangle_{\eta \in \mathcal{T}_\alpha}$ is spread out over A. Furthermore, $\langle b_{\trianglerighteq \eta^\frown \langle i \rangle} \rangle_{i \in \omega} \equiv_A \langle a_{\trianglerighteq \eta^\frown \langle i \rangle} \rangle_{i \in \omega}$ for each $\eta \in \mathcal{T}_\alpha$.

Modeling properties on ill-founded trees and its application

sketch of proof. If $\alpha = \omega$ then $\langle a_\eta \rangle_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over A we may assume $\alpha = n < \omega$ and $\eta \in \mathcal{T}_n$. We find a global A-invariant type $q \supseteq \mathrm{tp}(\langle b_{\trianglerighteq \eta^\frown \langle 0 \rangle} \rangle / A)$ s.t. $\langle b_{\trianglerighteq \eta^\frown \langle i \rangle} \rangle \models q_{\lceil A \langle b_{\trianglerighteq \eta^\frown \langle j \rangle} \rangle_{j<i}}$ for each $i \in \omega$. For each $\nu \neq \emptyset$, the proper sub-tree $\langle a_{\trianglerighteq \nu} \rangle$ is s-indiscernible. Thus the case of $\eta \neq \emptyset$ is clear. Assume $\eta = \emptyset$ and fix a formula φ arbitrary. Then

$$\bar{\chi} = \bigcup_{k \in \omega} \{ \langle i_k \rangle^\frown \chi : \chi \in \bar{\chi}_k \} \sim_s \{ \chi \in \mathcal{T}_n : \chi \trianglerighteq \langle i \rangle \text{ for some } i \in \omega \}$$

and $\langle b_{\trianglerighteq \langle i \rangle} \rangle_{i \in \omega} \equiv_{A,\varphi} \bar{a}_{\bar{\chi}}$, where $i_0 < i_1 < \ldots < \omega$ and $\bar{\chi}_k \sim_s \{ \chi : \chi \in \mathcal{T}_{n-1} \}$ for all $k \in \omega$. By the mutual s-indiscernibility of $\langle a_{\trianglerighteq \langle i \rangle} \rangle_{i \in \omega}$, we have $\langle b_{\trianglerighteq \langle i \rangle} \rangle_{i \in \omega} \equiv_{A,\varphi} \langle a_{\trianglerighteq \langle i_k \rangle} \rangle_{k \in \omega}$. By the order indiscernibility of $\langle a_{\trianglerighteq \langle i \rangle} \rangle_{i \in \omega}$, we have $\langle b_{\trianglerighteq \langle i \rangle} \rangle_{i \in \omega} \equiv_{A,\varphi} \langle a_{\trianglerighteq \langle i \rangle} \rangle_{i \in \omega}$. Since we choose φ arbitrary, we have $\langle b_{\trianglerighteq \langle i \rangle} \rangle_{i \in \omega} \equiv_A \langle a_{\trianglerighteq \langle i \rangle} \rangle_{i \in \omega}$ and hence $b_{\trianglerighteq \langle i \rangle} \models q_{\lceil A(b_{\trianglerighteq \langle j \rangle})_{j<i}}$ for all $i \in \omega$, where q is a global A-invariant type extending $\mathrm{tp}(a_{\trianglerighteq \langle 0 \rangle} / A)$. Therefore $\langle b_\eta \rangle_{\eta \in \mathcal{T}_n}$ is spread out over A.

Construction of an ATP structure which has no quantifier free SOP_2 witness

Definition [M. Džamonja and S. Shelah]

(i) $\varphi(x,y)$ is said to be having the *1-strong order property* (SOP_1) if there is a binary-tree-indexed $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ of parameters such that

- $\{\varphi(x, a_{\eta\restriction n})\}_{n\in\omega}$ is consistent for all $\eta \in {}^{\omega}2$,
- $\{\varphi(x, a_{\eta^\frown 1}), \varphi(x, a_{\eta^\frown 0^\frown \nu})\}$ is inconsistent for all $\eta, \nu \in {}^{\omega>}2$.

(ii) $\varphi(x,y)$ is said to be having the *2-strong order property* (SOP_2) if there is a binary-tree-indexed $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ of parameters such that

- $\{\varphi(x, a_{\eta\restriction n})\}_{n\in\omega}$ is consistent for all $\eta \in {}^{\omega}2$,
- $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu$,

Definition

We say $X \subseteq {}^{\omega>}2$ is antichain if $\eta \perp \nu$ for all $\eta, \nu \in X$.

(iii) $\varphi(x,y)$ is said to be having the *antichain tree property* (ATP) if there is a binary-tree-indexed $\langle a_\eta \rangle_{\eta \in {}^{\omega>}2}$ of parameters such that

- $\{\varphi(x, a_\eta)\}_{\eta\in X}$ is consistent for each antichain $X \subseteq {}^{\omega>}2$,
- $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \trianglelefteq \nu$,

Construction of an ATP structure which has no quantifier free SOP_2 witness

Remark

- $SOP_2 \Rightarrow SOP_1$.
- $ATP \Rightarrow SOP_1$.
- SOP_1 & $NSOP_2 \Rightarrow ATP$.
- It is not known whether SOP_1 implies SOP_2. If we assume a theory has SOP_1 but does not have SOP_2, then the theory has ATP.
- In the last year RIMS, we gave an example of theory which has ATP, witnessed by a formula whose finite conjunctions do not have SOP_2. By this example we can say that SOP_1 and SOP_2 are distinct at the level of formulas.

Proposition

There exists an ATP theory which has no quantifier free formula witnessing SOP_2.

Construction of an ATP structure which has no quantifier free SOP_2 witness

Let

α_i : the number of all maximal antichain in ${}^{i>}2$ for each $i \in \omega$,

β_i : the set of all maximal antichains in ${}^{i>}2$ for each $i \in \omega$,

μ_i : a bijection from α_i to β_i for each $i \in \omega$,

$A_i = \{a^i_l : l < \alpha_i\}$,

$B_i = \{b^i_\eta : \eta \in {}^{i>}2\}$,

$C_i = A_i \cup B_i$.

R : binary relation symbol,

$\mathcal{L} = \{R\}$.

For each $i \in \omega$, let $\mathcal{M}_i$ be an $\mathcal{L}$-structure such that $\mathbb{C}_i = (C_i; R^{\mathbb{C}_i})$, where

$R^{\mathbb{C}_i} = \{(a^i_l, b^i_\eta) \in A_i \times B_i \mid \eta \in \mu_i(l)\}$,

Then $Th(\mathbb{C}_i)$ has an antichain tree with height i, for all $i \in \omega$.

Construction of an ATP structure which has no quantifier free SOP_2 witness

For each $X \subseteq {}^{\omega>}2$ and $\eta \in {}^{\omega>}2$, let $\eta^\frown X$ denote $\{\eta^\frown \nu : \nu \in X\}$.

For each $n \in \omega$, let ε_n be a map from $\alpha_n \cup {}^{n>}2$ to $\alpha_{n+1} \cup {}^{n+1>}2$ which maps $x \mapsto x$ for all $x \in \alpha_n \cup {}^{n>}2$, and define $\varepsilon_n^* : C_n \to C_{n+1}$ by

$$a_l^n \mapsto a_{\varepsilon_n(l)}^{n+1} \quad \text{and} \quad b_\eta^n \mapsto b_{\varepsilon_n(\eta)}^{n+1}.$$

Then ε_n^* is an embedding. So we can regard $\mathbb{C}_n$ as a substructure of $\mathbb{C}_{n+1}$ with respect to ε_n^*.

Construction of an ATP structure which has no quantifier free SOP_2 witness

Let $\mathcal{P}_n = \{\Gamma_i\}_{i<2^{\alpha_n}}$ be the power set of α_n and p_n the cardinality of $\mathcal{P}_n$.

For each $n \in \omega$, let δ_n be a map from $\alpha_n \cup {}^{n>}2$ to $\alpha_{n+p_n} \cup {}^{n+p_n>}2$ which maps $\eta \mapsto \langle 0^{p_n} \rangle^\frown \eta$ for all $\eta \in {}^{n>}2$ and $l \mapsto \delta_n(l)$ for all $l \in \alpha_n$ where

$$\delta_n(l) = \mu_{n+p_n}^{-1} \left(\begin{array}{l} (\langle 0^{p_n} \rangle^\frown \mu_n(l) \\ \cup \{ \langle 0^{p_n - i - 1} \rangle^\frown \langle 1 \rangle : l \notin \Gamma_i \} \\ \cup \{ \langle 0^{p_n - i - 1} \rangle^\frown \langle 10 \rangle : l \in \Gamma_i \} \\ \cup \{ \langle 0^{p_n - i - 1} \rangle^\frown \langle 11 \rangle : l \in \Gamma_i \} \end{array} \right).$$

Define $\delta_n^* : C_n \to C_{n+p_n}$ by $a_l^n \mapsto a_{\delta_n(l)}^{n+p_n}$ and $b_\eta^n \mapsto b_{\delta_n(\eta)}^{n+p_n}$. Then δ_n^* is an embedding. So we can regard $\mathbb{C}_n$ as a substructure of $\mathbb{C}_{n+p_n}$ with respect to δ_n^*.

Construction of an ATP structure which has no quantifier free SOP_2 witness

Then we obtain a chain of $\mathcal{L}$-structures,

$$\mathbb{C}_1 \xrightarrow{\varepsilon_1^*} \mathbb{C}_2 \xrightarrow{\delta_2^*} \mathbb{C}_{2+p_2} \xrightarrow{\varepsilon_{2+p_2}^*} \cdots \mathbb{C}_n \xrightarrow{\varepsilon_n^*} \mathbb{C}_{n+1} \xrightarrow{\delta_{n+1}^*} \mathbb{C}_{n+1+p_{n+1}} \cdots .$$

We choose $\mathbb{C}'_n$ recursively as follow

$$\begin{aligned}
&\mathbb{C}'_1 = \mathbb{C}_1,\\
&\mathbb{C}'_2 = \mathbb{C}_2,\\
&\mathbb{C}'_{2i+1} = \mathbb{C}_{k+p_k} \text{ where } \mathbb{C}_k = \mathbb{C}'_{2i},\\
&\mathbb{C}'_{2i+2} = \mathbb{C}_{k+1} \text{ where } \mathbb{C}_k = \mathbb{C}'_{2i+1},
\end{aligned}$$

And for each $i \in \omega$, let $A'_i = A_k$ and $B'_i = B_k$ where $\mathbb{C}'_i = \mathbb{C}_k$. Let $\mathbb{C} = \bigcup_{i\in\omega} \mathbb{C}'_i$, $A = \bigcup_{i\in\omega} A'_i$, $B = \bigcup_{i\in\omega} B'_i$, and C be the universe of $\mathbb{C}$,

Construction of an ATP structure which has no quantifier free SOP_2 witness

Remark

- Either $\mathbb{C} \models \exists x R(x, c)$ or $\mathbb{C} \models \exists y R(c, y)$, for all $c \in C$.
- $\mathbb{C} \models \exists y R(c, y)$ if and only if $c \in A$.
- $\mathbb{C} \models \exists x R(x, c)$ if and only if $c \in B$.
- $R(x, y)$ forms an antichain tree in $\mathrm{Th}(\mathbb{C})$.

Construction of an ATP structure which has no quantifier free SOP_2 witness

Proposition

- *There is no* $b_0, b_1, b_2, b_3 \in B$ *such that*

$$\begin{aligned}\mathbb{C} \models \quad & b_0 \neq b_1 \wedge b_2 \neq b_3 \\ & \wedge \exists x(R(x, b_0) \wedge R(x, b_1)) \wedge \exists x(R(x, b_2) \wedge R(x, b_3)) \\ & \wedge \neg\exists x(R(x, b_0) \wedge R(x, b_2)) \wedge \neg\exists x(R(x, b_0) \wedge R(x, b_3)) \\ & \wedge \neg\exists x(R(x, b_1) \wedge R(x, b_2)) \wedge \neg\exists x(R(x, b_1) \wedge R(x, b_3)).\end{aligned}$$

- *For all* $b_0, ..., b_n \in B$, *if* $\mathbb{C} \models \neg\exists x(R(x, b_0) \wedge \cdots \wedge R(x, b_n))$, *then there exist* $i < j < n^+$ *such that* $\mathbb{C} \models \neg\exists x(R(x, b_i) \wedge R(x, b_j))$.

Construction of an ATP structure which has no quantifier free SOP_2 witness

Proposition

- *For all* $a_0, ..., a_n, a'_0, ..., a'_m \in A$,
  if $a_i \neq a'_j$ *for each* $i \in n^+$, $j \in m^+$,
  then $\mathbb{C} \models \exists y(R(a_0, y) \wedge ... \wedge R(a_n, y) \wedge \neg R(a'_0, y) \wedge ... \wedge \neg R(a'_m, y))$.

- *For all* $b_0, ..., b_n, b'_0, ..., b'_m \in B$,
  if $b_i \neq b'_j$ *for each* $i \in n^+, j \in m^+$,
  and $\mathbb{C} \models \exists x(R(x, b_i) \wedge R(x, b_j))$ *for all* i, j,
  then $\mathbb{C} \models \exists x(R(x, b_0) \wedge ... \wedge R(x, b_n) \wedge \neg R(x, b'_0) \wedge ... \wedge \neg R(x, b'_m))$.

Construction of an ATP structure which has no quantifier free SOP_2 witness

Proposition

The formula $\varphi(\bar{x}, \bar{x}', y) = \bigwedge_{i<n^+} R(x_i, y) \wedge \bigwedge_{j<m^+} \neg R(x'_j, y)$ does not witness SOP_2 modulo $\mathrm{Th}(\mathbb{C})$ for all $n, m \in \omega$.

sketch of proof. To get a contradiction we assume there exists $\langle \bar{c}^\eta, \bar{c}'^\eta \rangle_{\eta \in \omega^{>}2}$ which witnesses SOP_2 with $\varphi(\bar{x}, \bar{x}', y)$. We may assume $\langle \bar{c}_\eta \rangle_{\eta \in \omega^{>}2}$ is strongly indiscernible by the modeling property. Thus

$$\{R(c_0^{\langle 0 \rangle}, y), ..., R(c_n^{\langle 0 \rangle}, y), \neg R(c_0'^{\langle 0 \rangle}, y), ..., \neg R(c_m'^{\langle 0 \rangle}, y)$$
$$R(c_0^{\langle 1 \rangle}, y), ..., R(c_n^{\langle 1 \rangle}, y), \neg R(c_0'^{\langle 1 \rangle}, y), ..., \neg R(c_m'^{\langle 1 \rangle}, y)\}$$

is inconsistent. As we observed, there are $i \in n^+$ and $j \in m^+$ such that $c_i^{\langle 0 \rangle} = c_j'^{\langle 1 \rangle}$ or $c_i^{\langle 1 \rangle} = c_j'^{\langle 0 \rangle}$. We may assume $c_i^{\langle 0 \rangle} = c_j'^{\langle 1 \rangle}$. Then $c_i^{\langle 0 \rangle} = c_j'^{\langle 10 \rangle}$, $c_i^{\langle 0 \rangle} = c_j'^{\langle 11 \rangle}$ and $c_i^{\langle 10 \rangle} = c_j'^{\langle 11 \rangle}$ by strong indiscernibility. Thus $c_i^{\langle 10 \rangle} = c_j'^{\langle 11 \rangle} = c_i^{\langle 0 \rangle} = c_j'^{\langle 10 \rangle}$. But it is a contradiction because the set $\{R(c_0^{\langle 10 \rangle}, y), ..., R(c_n^{\langle 10 \rangle}, y), \neg R(c_0'^{\langle 10 \rangle}, y), ..., \neg R(c_m'^{\langle 10 \rangle}, y)\}$ is consistent.

Construction of an ATP structure which has no quantifier free SOP_2 witness

Proposition

The formula $\varphi(x, \bar{y}, \bar{y}') = \bigwedge_{i<n^+} R(x, y_i) \wedge \bigwedge_{j<m^+} \neg R(x, y'_j)$ does not witness SOP_2 modulo $Th(\mathbb{C})$ for all $n, m \in \omega$.

sketch of proof. As in the previous proposition, we assume there is a strongly indiscernible tree of parameters $\langle \bar{c}^{\eta}, \bar{c}^{'\eta} \rangle_{\eta \in \omega^{>2}}$ which witnesses SOP_2 with $\varphi(x, \bar{y}, \bar{y}')$. Then

$$\{R(x, c_0^{\langle 0 \rangle}), ..., R(x, c_n^{\langle 0 \rangle}), \neg R(x, c_0^{'\langle 0 \rangle}), ..., \neg R(x, c_m^{'\langle 0 \rangle})$$
$$R(x, c_0^{\langle 1 \rangle}), ..., R(x, c_n^{\langle 1 \rangle}), \neg R(x, c_0^{'\langle 1 \rangle}), ..., \neg R(x, c_m^{'\langle 1 \rangle})\}$$

is inconsistent. Then there are $i \in n^+$, $j \in m^+$ such that $c_i^{\langle 0 \rangle} = c_j^{'\langle 1 \rangle} \vee c_i^{\langle 1 \rangle} = c_j^{'\langle 0 \rangle}$, or there are $i, j \in n^+$ such that $\{R(x, c_i^{\langle 0 \rangle}), R(x, c_j^{\langle 1 \rangle})\}$ is inconsistent. If it is in the first case, then we can find a contradiction by using the same argument in the previous proposition.

Construction of an ATP structure which has no quantifier free SOP_2 witness

(continue).

So we assume $\{R(x, c_i^{\langle 0 \rangle}), R(x, c_j^{\langle 1 \rangle})\}$ is inconsistent for some $i, j \in n^+$.

By strong indiscernibility, $\{R(x, c_i^{\langle 0 \rangle}), R(x, c_j^{\langle 11 \rangle})\}$, $\{R(x, c_i^{\langle 00 \rangle}), R(x, c_j^{\langle 1 \rangle})\}$, and $\{R(x, c_i^{\langle 00 \rangle}), R(x, c_j^{\langle 11 \rangle})\}$ are inconsistent.

On the other hand, $\{R(x, c_i^{\langle 0 \rangle}), R(x, c_i^{\langle 00 \rangle})\}$ and $\{R(x, c_j^{\langle 1 \rangle}), R(x, c_j^{\langle 11 \rangle})\}$ are consistent by the path consistency condition. But then $c_i^{\langle 0 \rangle}, c_i^{\langle 00 \rangle}, c_j^{\langle 1 \rangle}, c_j^{\langle 11 \rangle}$ violate the first proposition in this subsection. $\square$

Construction of an ATP structure which has no quantifier free SOP_2 witness

Fact (A. Chernikov, N. Ramsey)

Let T be an arbitrary theory. If T has SOP_2 then there is a formula in a single free variable witnessing SOP_2, and vice versa.

Proposition

Let T be a theory and suppose $\varphi(x, \bar{y}) \vee \psi(x, \bar{z})$ witnesses SOP_2 modulo T, where x is a free variable, and $\bar{y}$, $\bar{z}$ are parameter variables. Then $\varphi(x, \bar{y})$ witnesses SOP_2 or $\psi(x, \bar{z})$ witnesses SOP_2.

Construction of an ATP structure which has no quantifier free SOP_2 witness

Proposition

In $\mathrm{Th}(\mathbb{C})$, there is no quantifier free formula witnessing SOP_2.

sketch of proof. Suppose φ be a quantifier free formula witnessing SOP_2 modulo $\mathrm{Th}(\mathbb{C})$. By the fact, we may assume φ is of the form $\varphi(z, \bar{w})$ where z is a single free variable, $\bar{w}$ is a parameter variable. By using disjunctive normal form, we may assume φ is a disjunction of conjunctions of basic formulas. Since the property that not having SOP_2 is closed under disjunction, we may assume φ is of the form

$$\bigwedge R(z, w_i) \wedge \bigwedge \neg R(z, w'_j) \wedge \bigwedge R(w''_k, z) \wedge \bigwedge \neg R(w'''_l, z).$$

But by the construction of $\mathbb{C}$, we have either $\mathbb{C} \models \exists y R(c, y)$ or $\mathbb{C} \models \exists x R(x, c)$ for all $c \in \mathbb{C}$. So φ is of the form either

$$\bigwedge R(z, w_i) \wedge \bigwedge \neg R(z, w'_j) \quad \text{or} \quad \bigwedge R(w_i, z) \wedge \bigwedge \neg R(w'_j, z).$$

As we observed, both of the above formulas can not witness SOP_2 modulo $\mathrm{Th}(\mathbb{C})$.

Construction of an ATP structure which has no quantifier free SOP_2 witness

But in $Th(\mathcal{M})$, a witness of SOP_2 exists. Let

$\varphi(x,y) := \neg\exists w(R(w,x) \wedge R(w,y)) \wedge \exists z(x \neq z \neq y \neq x \wedge \exists w(R(w,x) \wedge R(w,z)) \wedge \neg\exists w(R(w,y) \wedge R(w,z)))$. Then φ says "y is a predecessor of x in the set of parameters." (*i.e.*, $y \lhd x$) So, $\varphi(x,y)$ witnesses SOP_2.

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