One-variable theorem for the localized ATP

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Preliminaries on the tree properties

Definition

- By a *language*, we mean a set of constant symbols, relation symbols, and function symbols.
- Let $\mathcal{L}$ be a language. By an *$\mathcal{L}$-theory*, we mean a set of $\mathcal{L}$-sentences ($\mathcal{L}$-formula without free variable) which yields no contradiction.
- Let $\mathcal{L} = \{c_0, ..., R_0, ..., f_0, ...\}$ be a language, where c_i is a constant symbol for each i, R_i is an n_i-ary relation symbol for each i, and f_i is an m_i-ary function symbol for each i. By an *$\mathcal{L}$-structure (model)* $\mathbb{M}$, we mean a tuple $(M, c_0^{\mathbb{M}}, ..., R_0^{\mathbb{M}}, ..., f_0^{\mathbb{M}}, ...)$, where M is a set, $c_i^{\mathbb{M}} \in M$ for each i, $R_i^{\mathbb{M}} \subseteq M^{n_i}$ for each i, and $f_i : M^{m_k} \to M$ for each i. M is called the universe of $\mathbb{M}$.

Preliminaries on the tree properties

Definition

- Let $\mathcal{L}$ be a language, T be an $\mathcal{L}$-theory, σ be an $\mathcal{L}$-sentence, $\mathbb{M}$ be an $\mathcal{L}$-structure. By $\mathbb{M} \models \sigma$, we mean σ is true in $\mathbb{M}$. By $\mathbb{M} \models T$, we mean that $\mathbb{M} \models \sigma$ for all $\sigma \in T$.
- By a *type*, we mean a set of formulas which is consistent.
- For a formula $\varphi(x)$ and $m \in M$, if $\varphi(m)$ is true in M, then we write $M \models \varphi(m)$. For a type $p(x)$ and $m \in M$, if $\varphi(m)$ is true in M for all $\varphi(x) \in p(x)$, then we write $M \models p(m)$.

Remark

For a given theory T, we assume that there exists a sufficiently big saturated model $\mathbb{M}$ of T (which is called a monster model) such that:

- every small model M of T is an elementary substructure of $\mathbb{M}$,
- for every consistent type $p(x)$, there exists $m \in \mathbb{M}$ such that $\mathbb{M} \models p(m)$.

Preliminaries on the tree properties

Example

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ be the set of complex numbers, real numbers, rational numbers, integers, respectively.

- Let $\mathcal{L} = \{0, 1, +, -, \cdot\}$ be a language where 0, 1 are constant symbols, $+$, $-$, $\cdot$ are binary function symbols. Then $\mathbb{C}$ and $\mathbb{R}$ can be regarded as $\mathcal{L}$-structures and $\mathbb{C} \models \sigma$, $\mathbb{R} \models \neg\sigma$ where
$$\sigma := \exists x(x \cdot x = -1).$$
- Let $\mathcal{L} = \{<\}$ be a language where $<$ is a binary relation symbol. Then $\mathbb{Q}$ and $\mathbb{Z}$ can be regarded as $\mathcal{L}$-structures and $\mathbb{Q} \models \tau$, $\mathbb{Z} \models \neg\tau$ where
$$\tau := \forall xy(x < y \to \exists z(x < z < y)).$$

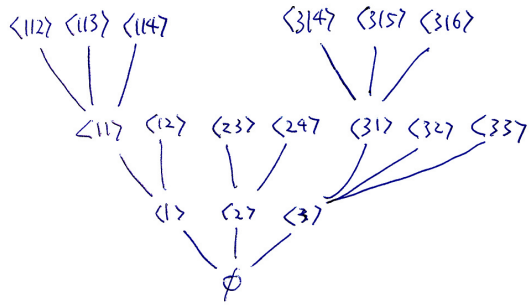
Preliminaries on the tree properties

Notation (Language of trees)

- $\omega^{<\omega}$ is the set of all finite sequences of natural numbers.
- ω^{ω} is the set of all infinite (countable) sequences of natural numbers.
- For $\eta, \nu \in \omega^{<\omega}$, by $\eta \trianglelefteq \nu$ we mean that η is an initial segment of ν.
- For $\eta, \nu \in \omega^{<\omega}$, by $\eta \perp \nu$ we mean that $\eta \not\trianglelefteq \nu$ and $\nu \not\trianglelefteq \eta$.
- We assume $\emptyset \in \omega^{<\omega}$ and $\emptyset \trianglelefteq \eta$ for all $\eta \in \omega^{<\omega}$.

Example

- $\langle 2 \rangle \trianglelefteq \langle 2, 3 \rangle \trianglelefteq \langle 2, 3, 1 \rangle \trianglelefteq \langle 2, 3, 1, 5 \rangle \trianglelefteq \ldots.$
- $\langle 7, 2, 5 \rangle \perp \langle 7, 2, 9 \rangle$.



Preliminaries on the tree properties

We can categorize first-order theories according to the combinatorial configurations. Let $\mathcal{L}$ be a language and T be an $\mathcal{L}$-theory.

Definition

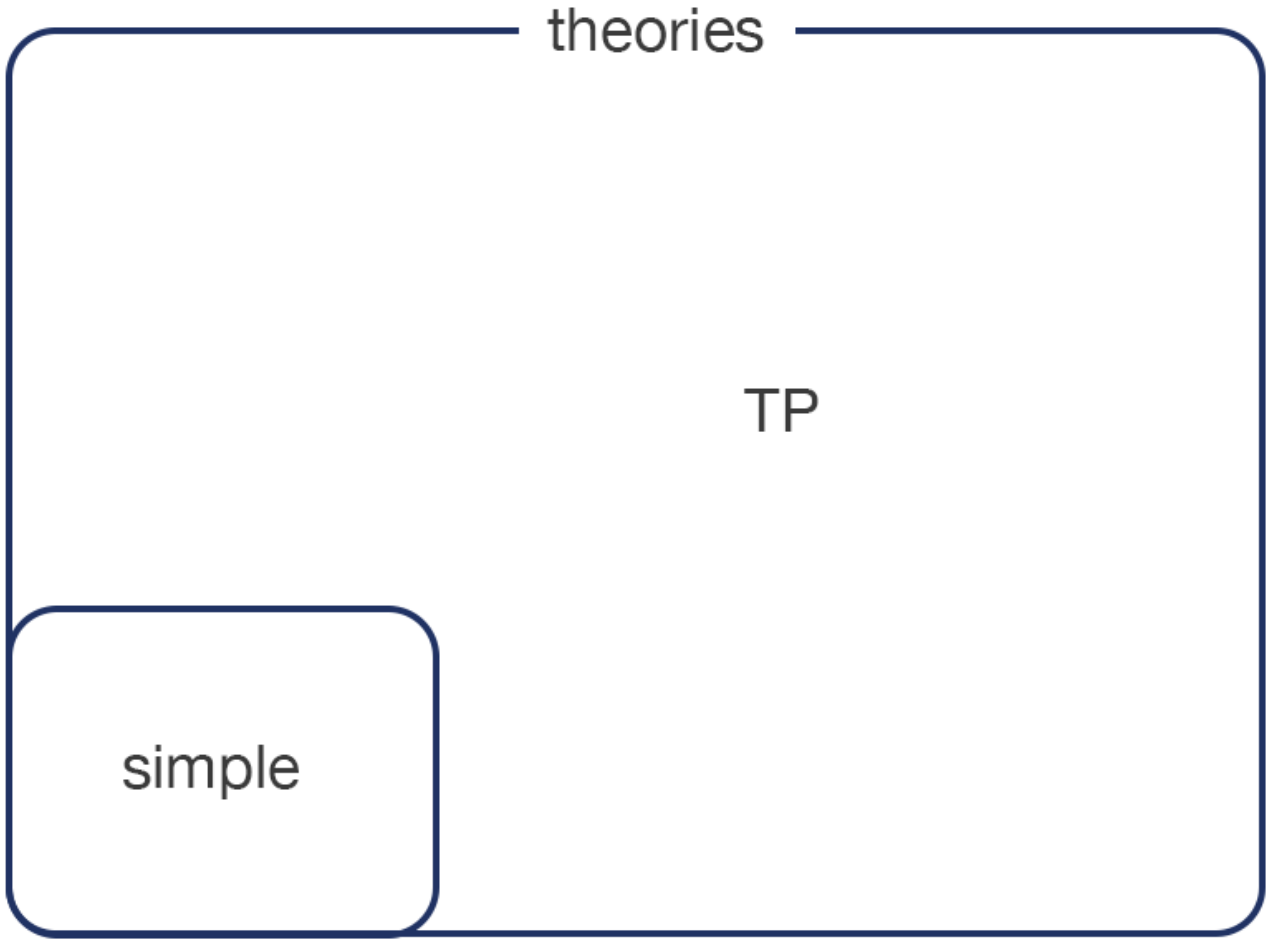
T is said to be *simple* if there is no $\mathcal{L}$-formula $\varphi(x, y)$, $\mathcal{L}$-structure $\mathbb{M} \models T$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

$\{\varphi(x, a_{\eta \restriction n})\}_{n \in \omega}$ is consistent for all $\eta \in \omega^\omega$,

$\{\varphi(x, a_{\eta^\frown i}), \varphi(x, a_{\eta^\frown j})\}$ is inconsistent for all $\eta \in \omega^{<\omega}$, $i < j \in \omega$.

If there exist φ, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in \omega^{<\omega}} \in \mathbb{M}$ satisfying the conditions above, then we say T has the *tree property (TP)*.

Preliminaries on the tree properties



Preliminaries on the tree properties

Definition

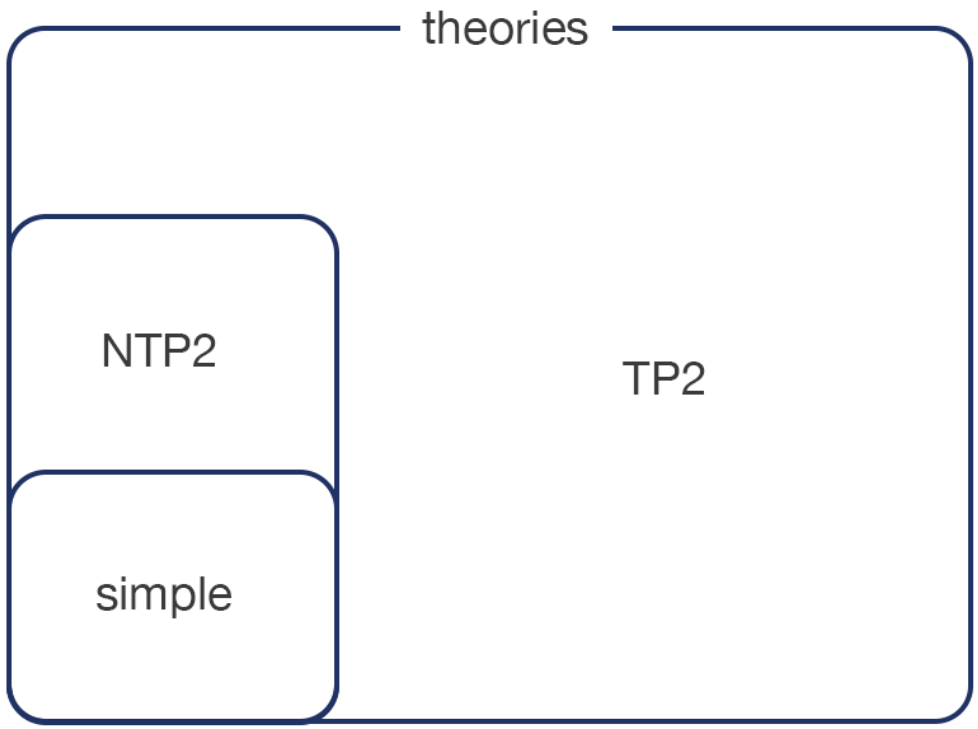
We say T has the *tree property of the second kind (TP_2)* if there exist $\varphi(x, y)$, $\mathbb{M} \models T$, $(a_{i,j})_{i,j \in \omega} \in \mathbb{M}$ such that

$\{\varphi(x, a_{n,f(n)})\}_{n \in \omega}$ is consistent for all $f : \omega \to \omega$,

$\{\varphi(x, a_{i,j}), \varphi(x, a_{i,k}))\}$ is inconsistent for all $i, j, k \in \omega$ with $j \neq k$.

We say T is *NTP_2* if it does not have TP_2.

Preliminaries on the tree properties



$$\mathrm{NTP}_2$$
$$\Uparrow$$
simple

Preliminaries on the tree properties

Definition

- We say T has the *tree property of the first kind (TP_1)* if there exist $\varphi(x,y)$, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in \omega^{<\omega}} \in \mathbb{M}$ such that

  $\{\varphi(x, a_{\eta \restriction n})\}_{n \in \omega}$ is consistent for all $\eta \in \omega^\omega$,
  $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu$.

  We say T is *NTP_1* if it does not have TP_1.
- We say T has the *2-strong order property (SOP_2)* if there exist $\varphi(x,y)$, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in 2^{<\omega}} \in \mathbb{M}$ such that

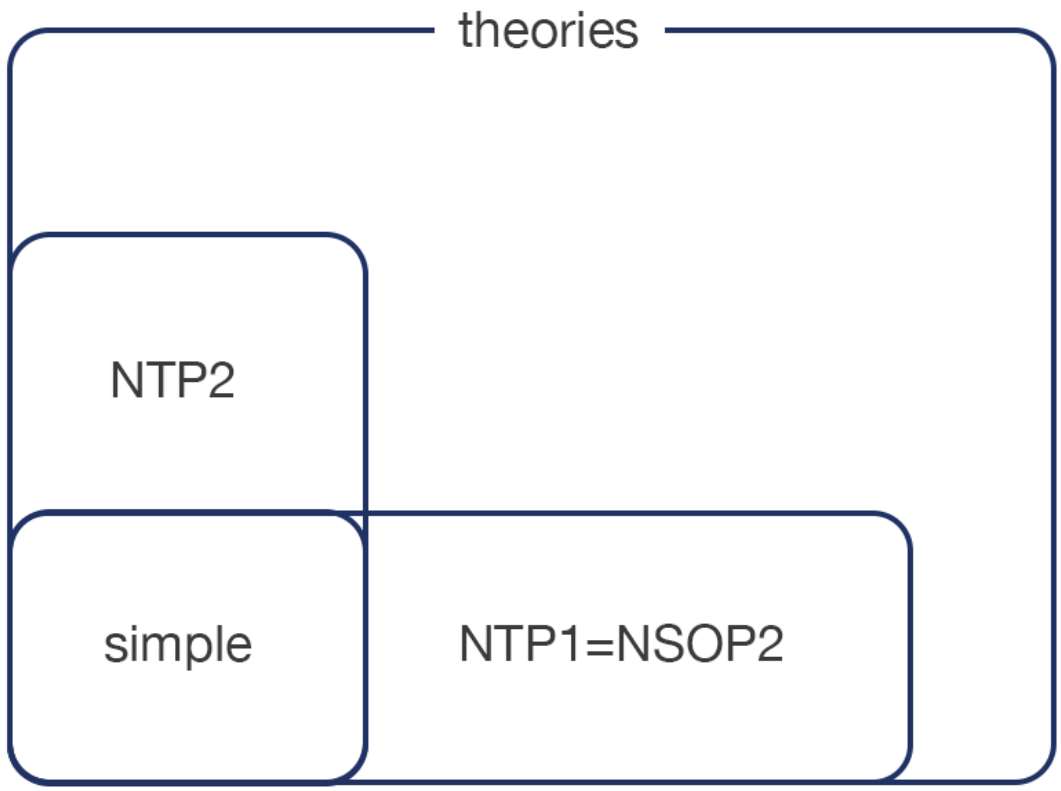
  $\{\varphi(x, a_{\eta \restriction n})\}_{n \in \omega}$ is consistent for all $\eta \in 2^\omega$,
  $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu$.

  We say T is *$NSOP_2$* if it does not have SOP_2.

Fact

- $\mathrm{TP}_1 \Leftrightarrow \mathrm{SOP}_2$.
- $\mathrm{TP} \Leftrightarrow \mathrm{TP}_1 \vee \mathrm{TP}_2$.

Preliminaries on the tree properties



$$\begin{array}{c} \mathrm{NTP}_2 \\ \Uparrow \\ \mathrm{simple} \Rightarrow \mathrm{NTP}_1 \Leftrightarrow \mathrm{NSOP}_2 \end{array}$$

Preliminaries on the tree properties

Definition

- We say T has the *1-strong order property (SOP_1)* if there exist $\varphi(x, y)$, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in 2^{<\omega}} \in \mathbb{M}$ such that

  $\{\varphi(x, a_{\eta \restriction n})\}_{n \in \omega}$ is consistent for all $\eta \in 2^\omega$,

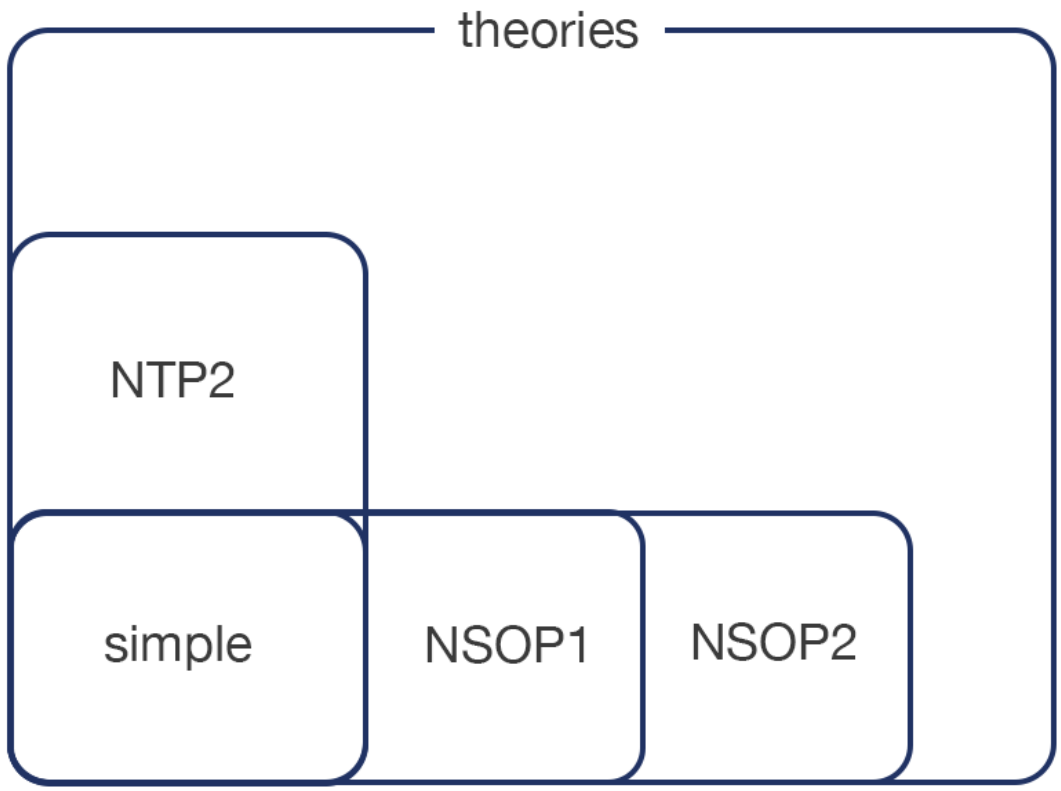
  $\{\varphi(x, a_{\eta^\frown 1}), \varphi(x, a_{\eta^\frown 0^\frown \nu})\}$ is inconsistent for all $\eta, \nu \in 2^{<\omega}$.

  We say T is *$NSOP_1$* if it does not have SOP_1.

Remark

- simple $\Rightarrow$ NSOP_1.
- $\mathrm{SOP}_2 \Rightarrow \mathrm{SOP}_1$ is well-known. We still do not know whether the converse holds.

Preliminaries on the tree properties



$$\begin{array}{ccccc} \mathrm{NTP}_2 & & & & \\ \Uparrow & & & & \\ \text{simple} & \Rightarrow & \mathrm{NSOP}_1 & \Rightarrow & \mathrm{NSOP}_2 \end{array}$$

Preliminaries on the tree properties

Definition

- $X \subseteq 2^{<\omega}$ is called an *antichain* if $\eta \perp \nu$ for all distinct $\eta, \nu \in X$.
- We say T has the *antichain tree property (ATP)* if there exist $\varphi(x, y)$, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in 2^{<\omega}} \in \mathbb{M}$ such that

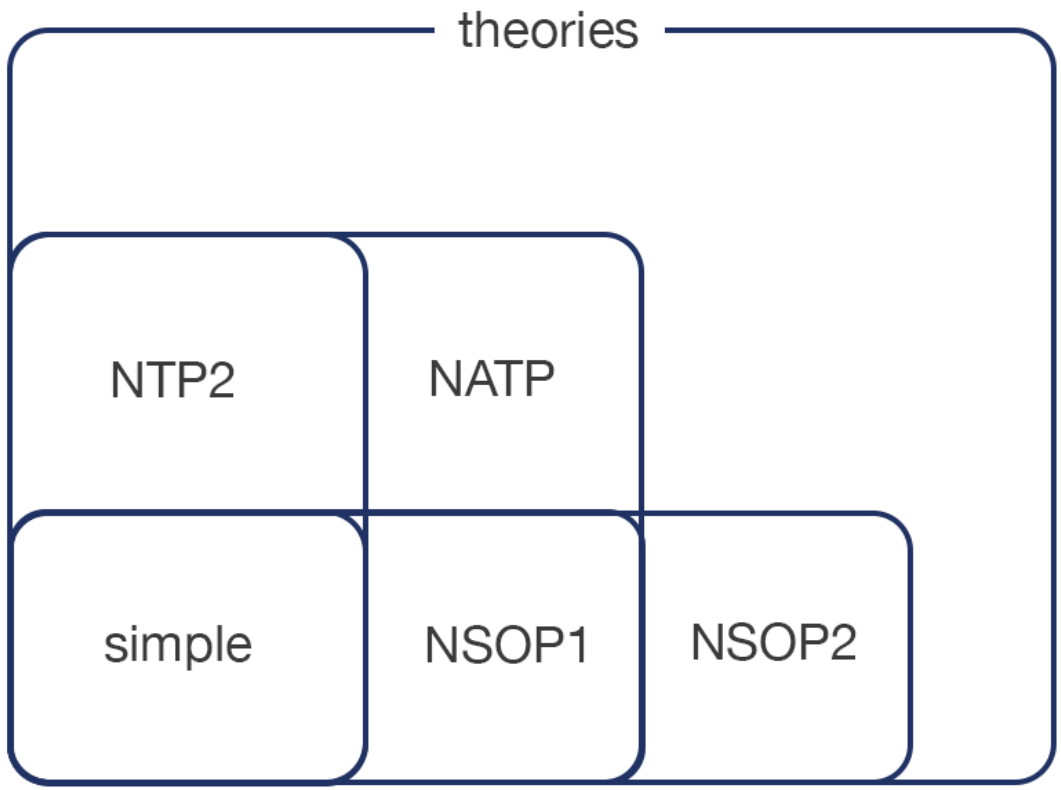
  $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for all antichain X in $2^{<\omega}$,
  $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \trianglelefteq \nu$.

  We say T is *NATP* if it does not have ATP.

Remark

- $\mathrm{NSOP}_1 \Rightarrow$ NATP.
- $\mathrm{NTP}_2 \Rightarrow$ NATP.
- If there exists a theory which is SOP_1 and NSOP_2, then the theory is ATP.

Preliminaries on the tree properties



$$\begin{array}{ccccc} \mathrm{NTP}_2 & \Rightarrow & \mathrm{NATP} & & \\ \Uparrow & & \Uparrow & & \\ \mathrm{simple} & \Rightarrow & \mathrm{NSOP}_1 & \Rightarrow & \mathrm{NSOP}_2 \end{array}$$

Preliminaries on the tree properties

Remark

- [NATP=NTP_2+$NSOP_1$] As a common extension of the class of NTP_2 theories and the class of $NSOP_1$, one can expect that theorems or properties which hold in NTP_2 and $NSOP_1$ theories also hold in NATP theories by using the similar argument used in NTP_2 and $NSOP_1$ cases.
- [ATP $\perp$ SOP_2] As the definition of ATP has an opposite nature to the definition of SOP_2, one can expect that theorems or properties which hold in $NSOP_2$ theories also hold in NATP theories by exchange the roles of paths and antichains in the tree indexed set in the argument used in $NSOP_2$ cases.

One-variable theorems for the tree properties

Remark

If a theory has TP_1 (or TP_2, SOP_1, ATP), then it is witnessed by a formula $\varphi(x, y)$ and the arity of x may vary. Thus if we want to show that a theory is NTP_1 (NTP_2, NSOP_1, NATP respectively) directly from the definition, then we need to check that there is no formula which witnesses TP_1 (TP_2, SOP_1, NATP).

But the complexity of formula increases rapidly as the arity of its free variable increases. This is the difficulty of showing a theory is NTP_1 (NTP_2, NSOP_1, NATP) directly from the definition. One-variable theorem for tree properties may simplify this problem.

One-variable theorems for the tree properties

One-variable theorem for TP_2 [A. Chernikov, 2012]

If T has TP_2, then it is witnessed by $\varphi(x, y)$ with $|x| = 1$.

One-variable theorem for TP_1 [A. Chernikov, N. Ramsey, 2016]

If T has TP_1, then it is witnessed by $\varphi(x, y)$ with $|x| = 1$.

One-variable theorem for SOP_1 [N. Ramsey, 2019]

If T has SOP_1, then it is witnessed by $\varphi(x, y)$ with $|x| = 1$.

One-variable theorem for ATP [J. Ahn, J. Lee, J. Kim, 2021]

If T has ATP, then it is witnessed by $\varphi(x, y)$ with $|x| = 1$.

When we check if a theory is NTP_1 (NTP_2, NSOP_1, NATP), then it is enough to show that every formula in a single free variable does not witnesses TP_1 (TP_2, SOP_1, NATP, resp.). Especially, even if the language is of many sorted, we do not need to check any formula whose variables are combination of multiple sorts.

Localized tree properties

Definition

A formula $\varphi(x, y)$ is said to have the SOP$_2$ inside a partial type q(x) if there are tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ in $\mathbb{M}$ such that

$q(x) \cup \{\varphi(x, a_{\eta \restriction n})\}_{n \in \omega}$ is consistent for all $\eta \in 2^\omega$,

$\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu$.

Fact [J. Dobrowolski, H. Kim, 2017]

One variable theorem for the localized SOP$_2$ holds. That is, if T has the SOP$_2$ inside of some partial type $q(x_0, ..., x_{n-1}) = \bigcup_{i<n} q_i(x_i)$, then T has the SOP$_2$ inside of $q_i(x_i)$ for some $i < n$.

Localized tree properties

Definition

A formula $\varphi(x, y)$ is said to have the ATP inside a partial type q(x) if there are tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ in $\mathbb{M}$ such that

$q(x) \cup \{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for each antichain X in $2^{<\omega}$,

$\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \trianglelefteq \nu$.

Lemma

The finite version of one variable theorem for the localized ATP holds. That is, if T has the ATP inside of $\{\rho(x_0, ..., x_{n-1})\}$ for some formula $\rho(x_0, ..., x_n) = \bigwedge_{i<n} \rho_i(x_i)$, then T has the ATP inside of $\{\rho_i(x_i)\}$ for some $i < n$.

Application

Let $\mathcal{L}$ be a language, T be an $\mathcal{L}$-theory, and H be an unary predicate symbol. For a model M of T, let $(M, H(M))$ denote an expansion of M to $\mathcal{L}_H$, where $H(M) = \{m \in M \mid \models H(m)\}$.

- We say $(M, H(M))$ is *dense codense* if every non-algebraic 1-type in $\mathcal{L}$ over any finite dimensional subset $A \subseteq M$ has realizations both in $H(M)$ and in $M \setminus \mathrm{acl}_T(A \cup H(M))$.
- A dense codencs expansion $(M, H(M))$ is called a *lovely pair* (*H-structure*) if $H(M)$ is an elementary $\mathcal{L}$-substructure (acl_T-independent subset, respectively) of M.

Application

If T is geometric complete theory, then it admit a lovely pair, and all lovely pairs have the same $\mathcal{L}_H$-theory, say T_P. Similarly, the same holds for H-structures. Let T^{ind} denote the theory of H-structures.

With other lemmas and observations, one-variable theorems allow us to prove the following:

Theorem [J. Dobrowolski, H. Kim, 2017]

If T is NSOP_2, then T_P and T^{ind} are also NSOP_2.

Theorem

If T is NATP, then T_P and T^{ind} are also NATP.

References

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[4] Jan Dobrowolski and Hyeungjoon Kim, A preservation theorem for theories without the tree property of the first kind, Mathematical Logic Quarterly, (2017).