Kim-forking and Kim-dividing in NATP theories

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Outline

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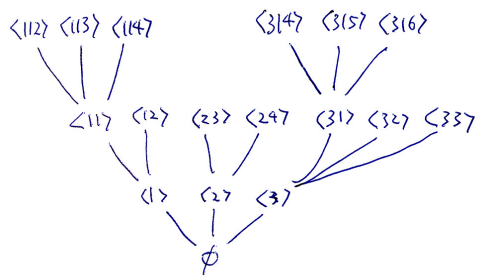
Preliminaries on the tree properties

Notation (Language of trees)

- $\omega^{<\omega}$ is the set of all finite sequences of natural numbers.
- ω^{ω} is the set of all infinite (countable) sequences of natural numbers.
- For $\eta, \nu \in \omega^{<\omega}$, by $\eta \trianglelefteq \nu$ we mean that η is an initial segment of ν.
- For $\eta, \nu \in \omega^{<\omega}$, by $\eta \perp \nu$ we mean that $\eta \not\trianglelefteq \nu$ and $\nu \not\trianglelefteq \eta$.
- We assume $\emptyset \in \omega^{<\omega}$ and $\emptyset \trianglelefteq \eta$ for all $\eta \in \omega^{<\omega}$.

Example

- $\langle 2 \rangle \trianglelefteq \langle 2, 3 \rangle \trianglelefteq \langle 2, 3, 1 \rangle \trianglelefteq \langle 2, 3, 1, 5 \rangle \trianglelefteq \ldots.$
- $\langle 7, 2, 5 \rangle \perp \langle 7, 2, 9 \rangle$.



Preliminaries on the tree properties

We can categorize first-order theories according to the combinatorial configurations. Let $\mathcal{L}$ be a language and T be an $\mathcal{L}$-theory.

Definition

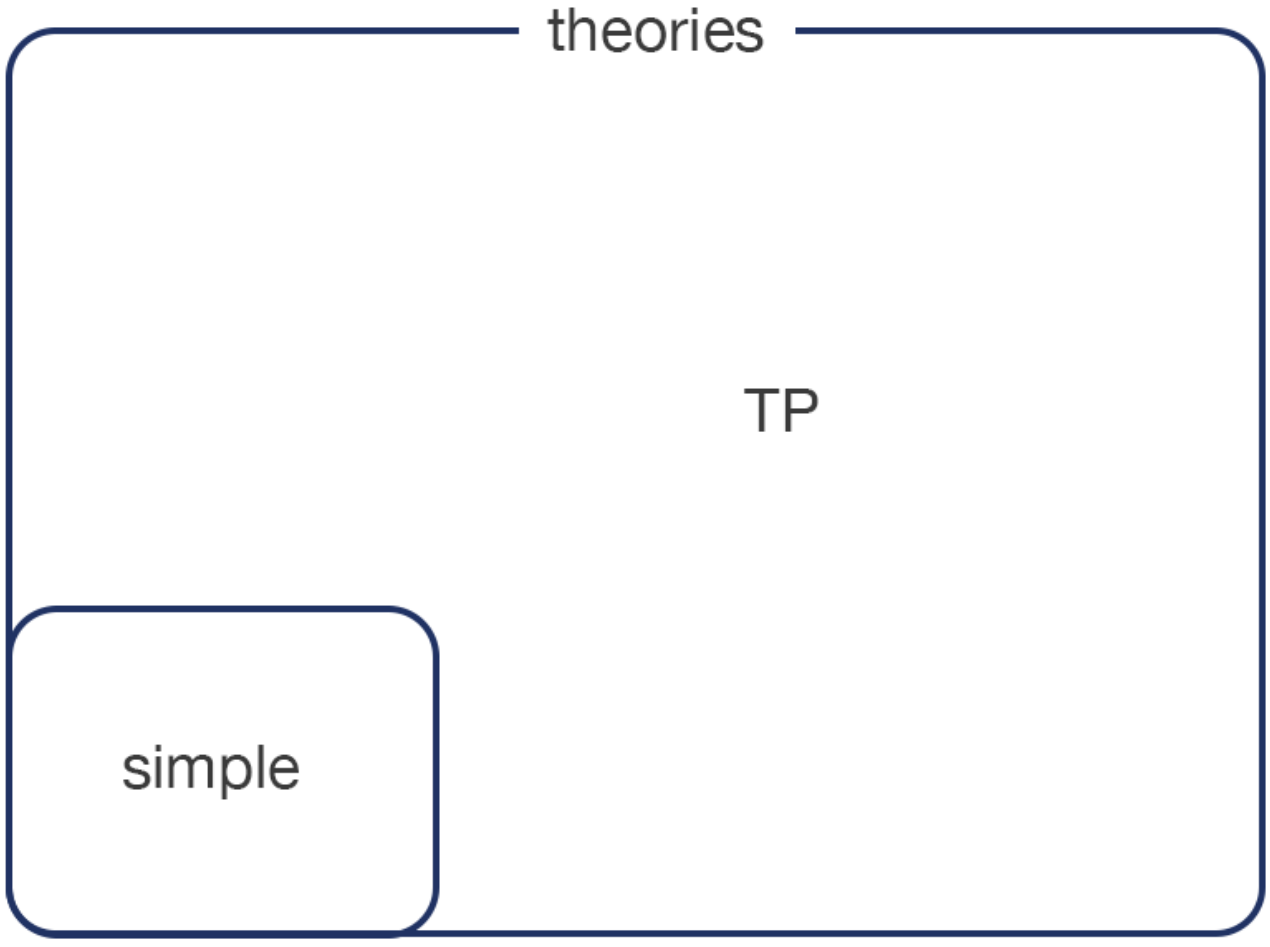
T is said to be *simple* if there is no $\mathcal{L}$-formula $\varphi(x, y)$, $\mathcal{L}$-structure $\mathbb{M} \models T$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

$\{\varphi(x, a_{\eta \restriction n})\}_{n \in \omega}$ is consistent for all $\eta \in \omega^\omega$,

$\{\varphi(x, a_{\eta^\frown i}), \varphi(x, a_{\eta^\frown j})\}$ is inconsistent for all $\eta \in \omega^{<\omega}$, $i < j \in \omega$.

If there exist φ, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in \omega^{<\omega}} \in \mathbb{M}$ satisfying the conditions above, then we say T has the *tree property (TP)*.

Preliminaries on the tree properties



Preliminaries on the tree properties

Definition

- We say T has the *1-strong order property (SOP_1)* if there exist $\varphi(x,y)$, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in 2^{<\omega}} \in \mathbb{M}$ such that

  $\{\varphi(x, a_{\eta \restriction n})\}_{n \in \omega}$ is consistent for all $\eta \in 2^\omega$,

  $\{\varphi(x, a_{\eta^\frown 1}), \varphi(x, a_{\eta^\frown 0^\frown \nu})\}$ is inconsistent for all $\eta, \nu \in 2^{<\omega}$.

  We say T is *$NSOP_1$* if it does not have SOP_1.

Definition

We say T has the *tree property of the second kind (TP_2)* if there exist $\varphi(x,y)$, $\mathbb{M} \models T$, $(a_{i,j})_{i,j \in \omega} \in \mathbb{M}$ such that

$\{\varphi(x, a_{n,f(n)})\}_{n \in \omega}$ is consistent for all $f : \omega \to \omega$,

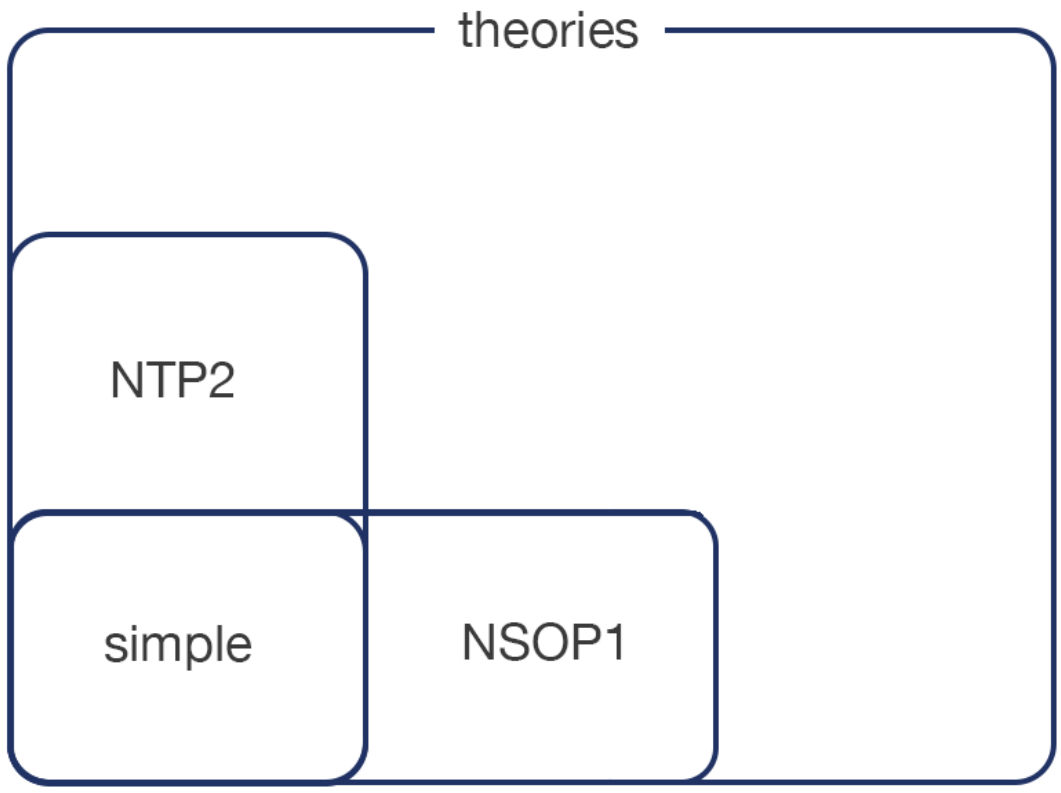
$\{\varphi(x, a_{i,j}), \varphi(x, a_{i,k}))\}$ is inconsistent for all $i, j, k \in \omega$ with $j \neq k$.

We say T is *NTP_2* if it does not have TP_2.

Remark

- simple $\Rightarrow$ NTP_2, simple $\Rightarrow$ NSOP_1.
- TP $\Leftrightarrow$ $\mathrm{SOP}_1 \vee \mathrm{TP}_2$.

Preliminaries on the tree properties



$$\begin{array}{c} \mathrm{NTP}_2 \\ \Uparrow \\ \mathrm{simple} \Rightarrow \mathrm{NSOP}_1 \end{array}$$

Preliminaries on the tree properties

Definition

- $X \subseteq 2^{<\omega}$ is called an *antichain* if $\eta \perp \nu$ for all distinct $\eta, \nu \in X$.
- We say T has the *antichain tree property (ATP)* if there exist $\varphi(x, y)$, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in 2^{<\omega}} \in \mathbb{M}$ such that

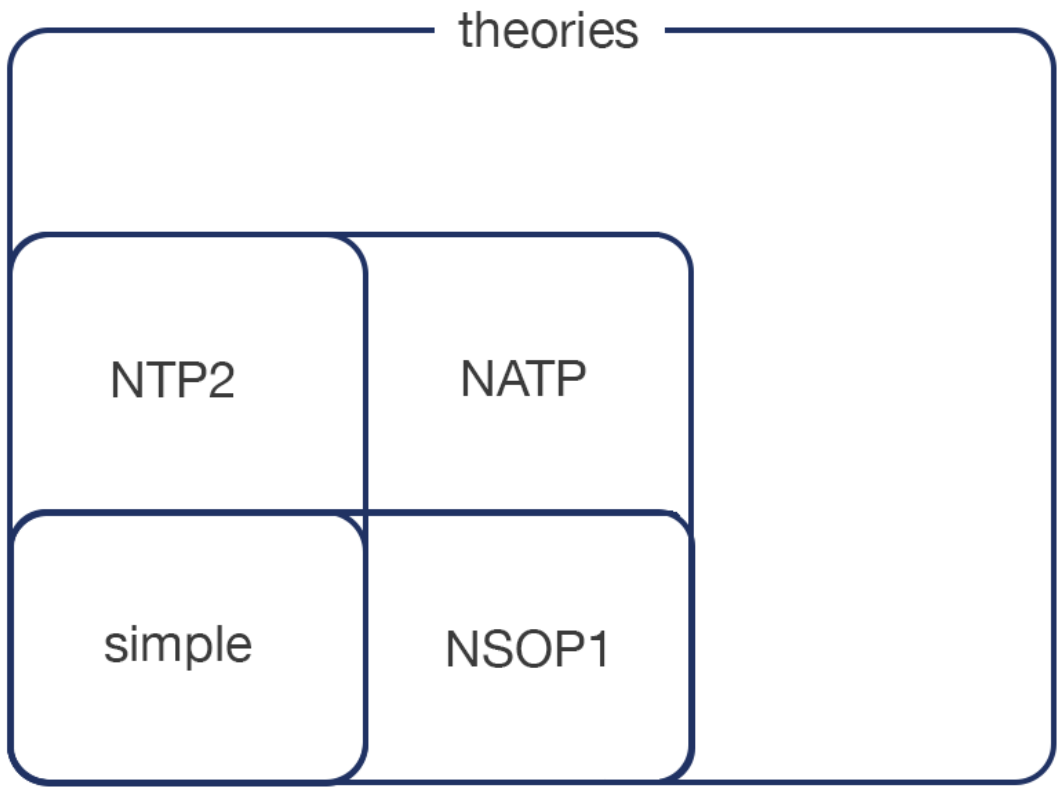
  $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for all antichain X in $2^{<\omega}$,
  $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \trianglelefteq \nu$.

  We say T is *NATP* if it does not have ATP.

Remark

- $\mathrm{NSOP}_1 \Rightarrow$ NATP.
- $\mathrm{NTP}_2 \Rightarrow$ NATP.

Preliminaries on the tree properties



$$\begin{array}{ccc} \mathrm{NTP}_2 & \Rightarrow & \mathrm{NATP} \\ \Uparrow & & \Uparrow \\ \mathrm{simple} & \Rightarrow & \mathrm{NSOP}_1 \end{array}$$

Kim-forking and Kim-dividing in NTP_2 and NSOP_1

Definition

Let $\mathcal{L}$ be a language, T be a complete $\mathcal{L}$-theory, $\mathbb{M} \models T$ be a moster model, and $A \subseteq \mathbb{M}$.

- We say a formula $\varphi(x, c)$ divides over A if there exists an indiscernible sequence $I = (c_i)_{i \in \omega}$ over A such that $\{\varphi(x, c_i)\}_{i \in \omega}$ is inconsistent.
- We say a partial type $p(x)$ forks over A if there exist $\psi_0(x, c_0), ..., \psi(x, c_n)$ such that $p(x) \vdash \bigvee_{i \leq n} \psi_i(x, c_i)$ and $\psi_i(x, c_i)$ divides over A for each $i \leq n$.

Fact [B. Kim, 1998]

In simple theories, forking and dividing are equivalent.
i.e., a formula forks over A iff it divides over A.

Kim-forking and Kim-dividing in NTP_2 and NSOP_1

Definition

Let $M \models T$ be a small model.

- A global type $p(x) \in S(\mathbb{M})$ is said to be invariant over M (M-invariant) if $\varphi(x, b) \leftrightarrow \varphi(x, b')$ for all $\varphi \in L$ and $b \equiv_M b'$.
- We say a formula $\varphi(x, c)$ Kim-divides over M if there exists a global M-invariant type p such that for all $I = (c_i)_{i \in \omega}$ with $c_i \models p|_{Mc_{<i}}$ for all $i \in \omega$, the set $\{\varphi(x, c_i)\}_{i \in \omega}$ is inconsistent.
- We say a partial type $p(x)$ Kim-forks over M if there exist $\psi_0(x, c_0), ..., \psi(x, c_n)$ such that $p(x) \vdash \bigvee_{i \leq n} \psi_i(x, c_i)$ and $\psi_i(x, c_i)$ Kim-divides over M for each $i \leq n$.

Fact [I. Kaplan, N. Ramsey, 2017]

In NSOP_1 theories, Kim-forking and Kim-dividing over a model are equivalent.
i.e., for a model M, a formula Kim-forks over M iff it Kim-divides over M.

Kim-forking and Kim-dividing in NTP_2 and NSOP_1

Remark

There is another notion of Kim-dividing as follows. It is known that two notions are coincide over a model in NSOP_1 theories. In this talk, we use the first notion of Kim-dividing.

Definition

Let $A \subseteq \mathbb{M}$ and $p(x) \in S(A)$.

- A sequence $I = (c_i)_{i \in \omega}$ is called a Morley sequence over A in p if it is indiscernible over A, $c_i \downarrow^f_A c_{<i}$ for all $i \in \omega$, and $c_i \models p$ for all $i \in \omega$.
- We say a formula $\varphi(x, c)$ Kim-divides over A if there exists a Morley sequence $I = (c_i)_{i \in \omega}$ over A in $\mathrm{tp}(c/A)$ such that $\{\varphi(x, c_i)\}_{i \in \omega}$ is inconsistent.

Kim-forking and Kim-dividing in NTP_2 and $NSOP_1$

Fact [A. Chernikov, I. Kaplan, 2012]

In NTP_2 theories, Kim-forking and Kim-dividing over a model are equivalent.

Remark

In fact, they proved that forking and dividing are equivalent over a model. But in their proof, if a formula forks over a model, then it Kim-divides over the model. (It was before the notion of Kim-forking and Kim-dividing appear.)

$$\begin{array}{ccc} \text{dividing} & \Leftarrow & \text{Kim-dividing} \\ \Downarrow & \nearrow\!\!\nearrow & \Downarrow \\ \text{forking} & \Leftarrow & \text{Kim-forking} \end{array}$$

Kim-forking and Kim-dividing in NTP_2 and $NSOP_1$

Definition

A relation $\downarrow$ is called a pre-independence relation if it is an invariant ternary relation. A pre-independence relation is standart if it satisfies the first 4 conditions below.

- [Monotonicity] if $aa' \downarrow_A bb'$, then $a \downarrow_A b$.
- [Base monotonicity] If $a \downarrow_A bc$, then $a \downarrow_{Ab} c$.
- [Transitivity on the left] If $a \downarrow_{Ab} c$ and $b \downarrow_A c$, then $ab \downarrow_A c$.
- [Right extension] If $a \downarrow_A b$, then for any c, there exists $c' \equiv_{Ab} c$ such that $a \downarrow_A bc'$.
- [Left extension] If $a \downarrow_A b$, then for any c, there exists $c' \equiv_{Ab} c$ such that $ac' \downarrow_A b$.

Definition

- We write $a \downarrow^f_A b$ if $\mathrm{tp}(A/Ab)$ does not fork over A
- We write $a \downarrow^K_A b$ if $\mathrm{tp}(A/Ab)$ does not Kim-fork over A
- We write $a \downarrow^i_A b$ if there exists an A-invariant global type p such that $p \supseteq \mathrm{tp}(a/Ab)$.

Kim-forking and Kim-dividing in NTP_2 and NSOP_1

Definition

- A global type $p(x) \in S(\mathbb{M})$ is called a coheir over A if for all $\varphi(x, c) \in q(x)$ there exists $a \in A$ such that $\models \varphi(a, c)$.
- A sequence $I = (c_i)_{i \in \omega}$ is called a coheir sequence over A if there exists a coheir p such that $c_i \models p|_{Ac_{<i}}$ for all $i \in \omega$. In this case we also say I is a coheir sequence of p over A.
- We write $a \mathop{\downarrow}^{u}_{A} b$ if there exists a global coheir p over A such that $p \supseteq \mathrm{tp}(a/Ab)$.

Remark

- If a global type is coheir over A, then it is invariant over A.
- $a \mathop{\downarrow}^{u}_{A} b \Rightarrow a \mathop{\downarrow}^{i}_{A} b \Rightarrow a \mathop{\downarrow}^{f}_{A} b \Rightarrow a \mathop{\downarrow}^{K}_{A} b$.
- $\mathop{\downarrow}^{u}$ satisfies monotonicity, base monotonicity, transitivity on the left, and right extension. Additionally, over a model, $\mathop{\downarrow}^{u}$ satisfies left extension (if M is a model and $a \mathop{\downarrow}_{M} b$, then for any c, there exists $c' \equiv_{Mb} c$ such that $ac' \mathop{\downarrow}_{M} b$).

Kim-dividing and coheir-dividing in NATP

Definition

Let A be a small set.

- We say a formula $\varphi(x, c)$ coheir-divides over A if there exists a global coheir p over A containing $\mathrm{tp}(c/A)$ such that for all $I = (c_i)_{i\in\omega}$ with $c_i \models p|_{Ac_{<i}}$ for all $i \in \omega$, the set $\{\varphi(x, c_i)\}_{i\in\omega}$ is inconsistent.
- We say a partial type $p(x)$ coheir-forks over A if there exist $\psi_0(x, c_0), ..., \psi(x, c_n)$ such that $p(x) \vdash \bigvee_{i\leq n} \psi_i(x, c_i)$ and $\psi_i(x, c_i)$ coheir-divides over A for each $i \leq n$.

$$\begin{array}{ccccc} \text{dividing} & \Leftarrow & \text{Kim-dividing} & \Leftarrow & \text{coheir-dividing} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{forking} & \Leftarrow & \text{Kim-forking} & \Leftarrow & \text{coheir-forking} \end{array}$$

Kim-dividing and coheir-dividing in NATP

Definition

We say a formula $\varphi(x, c)$ quasi-divides over a set A if there exists $(c_i)_{i<n}$ for some $n \in \omega$ such that $c_i \equiv_A c$ for all $i < n$ and $\{\varphi(x, c_i)\}_{i<n}$ is inconsistent.

Lemma (a special case of Broom Lemma)

If $\varphi(x, c) \vdash \bigvee_{i<n} \psi_i(x, d_i)$ and $\psi_i(x, d_i)$ coheir-divides over M for each $i < n$, then $\varphi(x, c)$ quasi-divides over M.

Definition

- We write $a \mathop{\smile}\limits^{ist}_{A} b$ if there exists an A-invariant global type p such that $p \supseteq \mathrm{tp}(a/Ab)$ and $B \mathop{\smile}\limits^{f}_{A} b$ for all $B \supseteq A$ and $b \models p|_B$.
- A global type $p(x) \in S(\mathbb{M})$ is called a strong coheir over A if p is a coheir over A and $B \mathop{\smile}\limits^{K}_{A} b$ for all $B \supseteq A$ and $b \models p|_B$.
- A sequence $I = (c_i)_{i\in\omega}$ is called a strong coheir sequence over A if there exists a strong coheir p such that $c_i \models p|_{Ac_{<i}}$ for all $i \in \omega$. In this case we also say I is a coheir sequence of p over A.

Kim-dividing and coheir-dividing in NATP

Strategy of proving Kim-forking=Kim-dividing in NTP_2

Assume T is NTP_2 and $M \models T$ is a model. Chernikov and Kaplan proved the equivalence of Kim-forking and Kim-dividing as follows.

1. If a formula divides over M, then it coheir-divides over M.
2. If a formula forks over M, then it quasi-divides over M by Bloom Lemma and 1.
3. By 2, every model is an extension base for $\downarrow^{ist}$. i.e., For any $c \in \mathbb{M}$, we have $c \downarrow^{ist}_{M} M$.
4. (Kim's Lemma in NTP_2) If a formula $\varphi(x, c)$ divides over M, then for all sequence $(c_i)_{i\in\omega}$ with $c_i \equiv_M c$ and $c_i \downarrow^{ist}_{M} c_{<i}$ for each $i \in \omega$, the set $\{\varphi(x, c_i)\}_{i\in\omega}$ is inconsistent.
5. By 3 and 4, if a formula forks over M, then it Kim-divides over M.

Thus, in fact, Chernikov and Kaplan proved that the equivalence of forking, dividing, Kim-forking, and Kim-dividing over a model in NTP_2.

Kim-dividing and coheir-dividing in NATP

Since forking and dividing are not equivalent in NSOP_1, we can not expect that forking implies coheir-dividing in NATP.

Question

Does Kim-forking imply coheir-dividing in NATP?

$$\begin{array}{ccccc} \text{dividing} & \Leftarrow & \text{Kim-dividing} & \Leftarrow & \text{coheir-dividing} \\ \Downarrow & & \Downarrow & ? & \Downarrow \\ \text{forking} & \Leftarrow & \text{Kim-forking} & \Leftarrow & \text{coheir-forking} \end{array}$$

Kim-dividing and coheir-dividing in NATP

Lemma [K., B. Lee, H. Lee]

In NATP, if a formula Kim-divides over a model, then it coheir-divides over the model.

$$\begin{array}{ccccc} \text{dividing} & \Leftarrow & \text{Kim-dividing} & \Leftrightarrow & \text{coheir-dividing} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{forking} & \Leftarrow & \text{Kim-forking} & \Leftrightarrow & \text{coheir-forking} \end{array}$$

Corollary

In NATP, if a formula Kim-forks over a model, then it quasi-divides over the model.

Corollary

In NATP, every model is an extension base for strong coheir. i.e., for all model M and $c \in \mathbb{M}$, there exists a strong coheir over M containing $\mathrm{tp}(c/M)$.

Kim-dividing and coheir-dividing in NATP

Definition

Let α be an ordinal. $\mathcal{T}_\alpha$ is the set of functions which satisfies:

(i) $\mathrm{dom}(f)$ is an end-segment of α of the form $[\beta, \alpha)$ for β is equal to 0 or a successor ordinal. If α is successor, we allow $\beta = \alpha$.

(ii) $\mathrm{ran}(f) \subseteq \omega$,

(iii) [finite support] there are only finitely many elements in its domain whose image is nonzero.

In particular, $\mathcal{T}_\omega$ is a tree which has no least element, whose paths are countable.

Kim-dividing and coheir-dividing in NATP

sketch of proof

Suppose $\varphi(x, c)$ Kim-divides over a model M, with M-invariant global type $p(x) \supseteq \mathrm{tp}(c/M)$ and assume that there is no coheir extending $\mathrm{tp}(c/M)$ witnessing coheir-dividing with $\varphi(x, c)$. Note that the number of all coheir extension of $\mathrm{tp}(c/M)$ is bounded and $\downarrow^u$ satisfies left extension over M. Let $\{q_i\}_{i<\lambda}$ be an enumeration of all global coheirs containing $\mathrm{tp}(c/M)$.
Then For all $A \supseteq M$ and an ordinal α, there exists $\{b_\eta\}_{\eta \in \mathcal{T}_\alpha}$ such that

(i) $b_\eta \models p|_{Mb_{\triangleright \eta}}$ for each $\eta \in \mathcal{T}_\alpha^2$,

(ii) For each $\eta \in \mathcal{T}_\alpha^2$, there exists $i < \theta$ such that $b_\eta \models q_i|_{Ab_{\perp_{lex}\eta}}$ and $b_\eta \not\models q_{i'}|_{Ab_{\perp_{lex}\eta}}$ for all $i' < i$,

(iii) $A' \downarrow^u_M A$ where $A' = \{b_\eta\}_{\eta \in \mathcal{T}_\alpha}$,

where $\mathcal{T}_\alpha$ is an ill-founded tree. Then by using colouring argument, compactness, automorphisms, and the ill-foundedness of $\mathcal{T}_\alpha$, we can construct an antichain tree.

Kim-dividing and coheir-dividing in NATP

Sufficient conditions of Kim-forking=kim-dividing in NATP over a model

1. (Kim's Lemma for NATP over a model) If $\varphi(x, c)$ Kim-divides over M then for all strong coheir sequence $(c_i)_{i\in\omega}$ with $c_0 = c$, the set $\{\varphi(x, c_i)\}_{i\in\omega}$ is inconsistent.
2. The existence of a global type, which is stronger than strong coheirs. i.e. For a model M and $c \in \mathbb{M}$, there exists a global coheir $p \supseteq \mathrm{tp}(c/M)$ such that $B \overset{i}{\downarrow}_M b$ for all $B \supseteq M$ and $b \models p|_B$.

References

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