Dense/Codense expansions on vector spaces

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October 19, 2022

Outline

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Languages, theories, structures

Definition

- By a *language*, we mean a set of constant symbols, relation symbols, and function symbols.
- Let $\mathcal{L}$ be a language. By an *$\mathcal{L}$-theory*, we mean a set of $\mathcal{L}$-sentences ($\mathcal{L}$-formula without free variable) which yields no contradiction.
- Let $\mathbb{M}$ be a mathematical structure, $\mathcal{L}$ a language. $Th_{\mathcal{L}}(\mathbb{M})$ is an $\mathcal{L}$-theory, the set of all $\mathcal{L}$-sentences that is true in $\mathbb{M}$.

Example

Let $\mathbb{R}$ be the set of real numbers. $\mathcal{L} = \{0, 1, +, -, \cdot, ^{-1}\}$.
Then $\mathrm{Th}_{\mathcal{L}}(\mathbb{R}) = \{\neg\exists x(x \cdot x = -1),$
$\forall y_0, y_1, y_2 \exists x(x \cdot x \cdot x + y_2 \cdot x \cdot x + y_1 \cdot x + y_0 = 0),$
$\cdots\}$

For a give structure and a language, we can consider the theory of the structure with respect to the language.

Languages, theories, structures

Conversely, for a given language $\mathcal{L}$ and an $\mathcal{L}$-theory, say T, we can consider every possible mathematical structure which satisfies all sentences in T.

Example

Let $\mathcal{L} = \{<\}$, where $<$ is a binary relation. Let T be an $\mathcal{L}$-theory given by

- $\forall x(x \not< x)$,
- $\forall x, y(x < y \wedge y < x \to x = y)$,
- $\forall x, y, z(x < y \wedge y < z \to x < z)$,
- $\forall x, y(x < y \vee y < x)$,
- $\forall x \exists y(x < y)$, $\forall x \exists y(y < x)$
- $\forall x, y \exists z(x < z < y)$,

Then there are numerously many structures satisfies T. For example, $\mathbb{R}$, $\mathbb{Q}$, $\oplus_{i<\omega}\mathbb{N}_i$, and so on. (This T is called DLO (dense linear order))

By considering T, we investigate common properties of structures of T.

Definable sets and classifying theories

Definition

- For a given structure $\mathbb{M}$, we say $X \subset \mathbb{M}$ is *$\mathcal{L}$-definable* if there exists an $\mathcal{L}$-formula $\varphi(x)$ such that $m \in X$ if and only if $\varphi(m)$ holds in $\mathbb{M}$. (X is defined by $\varphi(x)$.)
- For $\mathbb{M}$ and $b \in M$, we say $X \subset \mathbb{M}$ is *$\mathcal{L}$-definable over b* if there exists an $\mathcal{L}$-formula $\varphi(x, y)$ such that $m \in X$ if and only if $\varphi(m, b)$ holds in $\mathbb{M}$. (X is defined by $\varphi(x, b)$. Here, we say b a *parameter*.)

Example

For $\mathbb{R}$ and $\mathcal{L} = \{<\}$, every interval is definable. ($\varphi(x, y, z) := x < y < z$)
But the set of natural numbers is not definable in $\mathbb{R}$ with respect to $\mathcal{L}$.

If a language becomes richer, then there will be many more definable sets.

Definable sets and classifying theories

Considering possibility of the existence of a certain set of definable sets, we can classify theories.

Example

Q: For a structure of T, is it possible to find a set of definable sets $\{\varphi(x, a_\eta) : \eta$ is a node in the binary tree $2^{<\omega}\}$ such that

- $\varphi(x, a_{\eta_0}), ..., \varphi(x, a_{\eta_n})$ have a common solution where $\eta_0, ..., \eta_n$ lie in the same path in the binary tree.
- $\varphi(x, a_\eta), \varphi(x, a_\nu)$ have no common solution whenever η and ν are not lying in the same path?

If the answer is 'yes', then we say T is SOP_2, and if the answer is 'no', then T is said to be NSOP_2.

Example

In DLO, we can find a *witness* for SOP_2. Consider the formula $\varphi(x, y, z) := x < y < z$. So DLO is SOP_2.

Definable sets and classifying theories

There are many of model-theoretic properties which classify theories such as stable, simple, NIP, NTP_1, NTP_2, NSOP_1, NSOP_2, NSOP_3, ...

Definition

We say a theory T is NATP if it is *impossible* to find a set of definable sets $\{\varphi(x, a_\eta) : \eta \text{ is a node in the binary tree } 2^{<\omega}\}$ such that

(i) $\varphi(x, a_{\eta_0}), ..., \varphi(x, a_{\eta_n})$ have a common solution where η_i, η_j do not lie in the same path in the binary tree.

(ii) $\varphi(x, a_\eta), \varphi(x, a_\nu)$ have no common solution whenever η and ν are lying in the same path.

$$\begin{array}{ccccccc} \mathrm{NIP} & \Rightarrow & \mathrm{NTP}_2 & \Rightarrow & \mathrm{NATP} & & \\ \Uparrow & & \Uparrow & & \Uparrow & & \\ \text{stable} & \Rightarrow & \text{simple} & \Rightarrow & \mathrm{NTP}_1 & \Rightarrow \mathrm{NSOP}_3 & \Rightarrow \cdots \end{array}$$

Preservation of model-theoretic properties

Even if a theory has NATP (or has some other 'negative' property such as NTP_1, NIP, NSOP,...), the property can be disappeared when we add a new language to the given language. If we increase the language, then we have more many definable sets, so it may happen that the set of definable sets appears, which satisfies (i) and (ii) in the previous page.

One of our main interesting is that:
When does a given 'negative' model-theoretic property still remain after adding a new language to the given language?

In a joint work with JinHoo Ahn, Hyoyoon Lee, and Junguk Lee we found some conditions which preserve NATP after adding a new language.

Dense/codense expansion on vector spaces

$\mathbb{F}$: a field with characteristic 0
R: a subring of $\mathbb{F}$
$\mathcal{L}_0 = \{+, 0, \{\lambda\}_{\lambda\in\mathbb{F}}\}$: a language of vector spaces over $\mathbb{F}$.
$\mathcal{L}_{R\text{-mod}} = \{+, 0, \{r\}_{r\in R}\}$: a language of R-module
T_0: an $\mathcal{L}$-theory which is a theory for vector space over $\mathbb{F}$
(the set of all $\mathcal{L}_0$-sentences which is true in any vector space over $\mathbb{F}$)
T eliminates $\exists^\infty$, $\mathrm{acl} = \mathrm{span}_{\mathbb{F}}$.

Let G be a new unary relation symbol.
For an $\mathcal{L}_{R\text{-mod}}$-formula $\theta(\bar{x})$, let $P_\theta(\bar{x})$ be a new relation symbol.
Consider the extended language
$\mathcal{L}_G = \mathcal{L}_0 \cup \{G\} \cup \{P_\theta : \theta \text{ is an } \mathcal{L}_{R\text{-mod}}\text{-formula}\}$

Dense/codense expansion on vector spaces

We extend T_0 to an $\mathcal{L}_G$-theory T_G as follows:

(A) G is an R-submodule of the universe and for each $\mathcal{L}_{R\text{-mod}}$-formula $\theta(\bar{x})$ and $\bar{a}$, $P_\theta(\bar{a})$ if and only if $\bar{a} \in G$ and $G \models_{R\text{-mod}} \theta(\bar{a})$.

(B) Denote $\hat{R}$ by the field of fractions of R. If $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ are $\hat{R}$-linearly independent, then for all $g_1, \ldots, g_n \in G$,

$$\lambda_1 g_1 + \cdots + \lambda_n g_n = 0 \text{ implies } \bigwedge_i g_i = 0.$$

(C) For each $\bar{a}$, r, and an $\mathcal{L}_0$-formula $\varphi(x, \bar{y})$, if $r \in R \setminus \{0\}$ and there are infinitely many solutions of $\varphi(x, \bar{a})$, then $\varphi(x, \bar{a}) \wedge rG(x)$ has a solution.

(D) For each $\mathcal{L}_0$-formulas $\varphi(x, \bar{y})$, $\psi(x, \bar{y}, \bar{z})$, and $n \geq 1$,
$\forall \bar{y} \Big(\exists^\infty x \varphi(x, \bar{y}) \wedge \forall \bar{z} \exists^{\leq n} x \psi(x, \bar{y}, \bar{z})$
$\rightarrow \exists x \big(\varphi(x, \bar{y}) \wedge \forall z (G(\bar{z}) \rightarrow \neg \psi(x, \bar{y}, \bar{z})) \big) \Big)$.

Dense/codense expansion on vector spaces

(A) The set of solutions of $G(x)$ will be an R-submodule.

(B) If $\lambda_1, ..., \lambda_n \in \mathbb{F}$ are $\hat{R}$-linearly independent, then they are G-linearly independent.

(C) rG is dense in the universe for all $r \in R \setminus \{0\}$. (with respect to every L_0-definable set which has infinitely many solutions. Note that $rG \subseteq G$ but rG is not G in general)

(D) If $\varphi(x, \bar{a})$ has infinitely many solutions, $\psi(x, \bar{a}, \bar{b})$ has at most n-many solutions for every b, then there exists a solution m of $\varphi(x, \bar{a})$ such that $G \cap \psi(m, \bar{a}, z) = \emptyset$.
(The intersection between the set defined by $G(x)$ and the set defined by $\psi(m, \bar{a}, z)$ is empty.)

Dense/codense expansion on vector spaces

Theorem [Berenstein, d'Elbée, and Vassiliev, 2022]

If T_0 is NIP (or NTP_1, NTP_2, NSOP_1), then T_G is also NIP (or NTP_1, NTP_2, NSOP_1).

Theorem [Ahn, K, Lee, and Lee, 2022]

If T_0 is NATP, then T_G is also NATP.

Dense/codense expansion on vector spaces

In fact, we can repeat adding a unary relation.

$\mathbb{F}$: a field with characteristic 0
R: a subring of $\mathbb{F}$
$\mathcal{L}_0 = \{+, 0, \{\lambda\}_{\lambda \in \mathbb{F}}\}$: a language of vector spaces over $\mathbb{F}$.
$\mathcal{L}_{R\text{-mod}} = \{+, 0, \{r\}_{r \in R}\}$: a language of R-module
T_0: an $\mathcal{L}$-theory which is a theory for vector space over $\mathbb{F}$

$\mathcal{L} \supseteq \mathcal{L}_0$.
T: an $\mathcal{L}$-theory such that $T|_{\mathcal{L}_0} = T_0$
T eliminates $\exists^\infty$, $\mathrm{acl} = \mathrm{span}_{\mathbb{F}}$.
$\mathcal{L}_G = \mathcal{L} \cup \{G\} \cup \{P_\theta : \theta \text{ is an } \mathcal{L}_{R\text{-mod}}\text{-formula}\}$
Then we can extend T to an $\mathcal{L}_G$-theory T_G as above.

Theorem [Berenstein, d'Elbée, and Vassiliev, 2022]

If T_0 is NIP (or NTP_1, NTP_2, NSOP_1), then T_G is also NIP (or NTP_1, NTP_2, NSOP_1).

Theorem [Ahn, K, Lee, and Lee, 2022]

If T is NATP, then T_G is also NATP.

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