Pre-independence relations and forking in classification theory

Joonhee Kim
(joint work with Byunghan Kim and Hyoyoon Lee)

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In short, model theory is a study of mathematical structures in terms of language and definable sets.

Definition

- By a *language*, we mean a set of constant symbols, relation symbols, function symbols, and variables. (These symbols are called non-logical symbols. $\neg$(not), $\wedge$(and), $\vee$(or), $\exists$('there exists'), $\forall$('for all') are called logical symbols.)
- Let $\mathcal{L}$ be a language. We define *$\mathcal{L}$-terms* in a recursive way:
  - each constant symbol or variable is an $\mathcal{L}$-term,
  - $f(t_0, ..., t_n)$ is an $\mathcal{L}$-term for a given function symbol $f \in \mathcal{L}$ and $\mathcal{L}$-terms $t_0, ..., t_n$.
- Let $\mathcal{L}$ be a language. We define *$\mathcal{L}$-formula* in a recursive way:
  - $R(t_0, ..., t_n)$ is an $\mathcal{L}$-formula for a given relation symbol $R \in \mathcal{L}$ and $\mathcal{L}$-terms $t_0, ..., t_n$.
  - $\neg\varphi(\bar{x})$, $\varphi(\bar{x}) \wedge \psi(\bar{x})$, $\varphi(\bar{x}) \vee \psi(\bar{x})$ are $\mathcal{L}$-formulas for a given $\mathcal{L}$-formulas φ and ψ.
  - $\exists x\varphi(x, \bar{y})$ and $\forall x\varphi(x, \bar{y})$ are $\mathcal{L}$-formulas for a given $\mathcal{L}$-formula $\varphi(x, \bar{y})$.

Example

Let $\mathcal{L} = \{0, 1, +, -, \cdot, ^{-1}\}$ be a language for fields.
Then for each $f(\bar{x}), f_0(\bar{x}), ..., f_n(\bar{x}) \in \mathbb{Z}[\bar{x}]$,

- $f(\bar{x}), f_0(\bar{x}), ..., f_n(\bar{x})$ are $\mathcal{L}$-terms.
- $f(\bar{x}) = 0$ and $f(\bar{x}) = y$ are $\mathcal{L}$-formulas. (for the convenience we assume the language contains the equality symbol '$=$' implicitly. '$=$' is a binary relation symbol.)
- $\exists \bar{x}(f(\bar{x}) = 0)$ is an $\mathcal{L}$-formula.

Definition

- For a given $\mathcal{L}$-formula φ, if a variable x is in φ and is not captured by any quantifier, then we say *x is free in φ*.
  (x is not free in $\exists x(f(x) = y)$, but y is free.)
- An $\mathcal{L}$-formula without free variable is called an *$\mathcal{L}$-sentence*.
  (e.g. $\exists x(f(x) = 0)$)
- A set of $\mathcal{L}$-sentences is called an *$\mathcal{L}$-theory* if it does not yield a contradiction. Equivalently, if there is a mathematical structure such that every sentence of the set is true in the structure, then we say the set is an $\mathcal{L}$-theory.

Example

Let $\mathcal{L} = \{0, 1, +, -, \cdot, ^{-1}\}$ and ACF be a set of $\mathcal{L}$-sentences that consists of

(i) $\forall x[(x + 0 = x) \wedge (x \cdot 1 = x) \wedge (x \cdot 0 = 0)]$,

(ii) $\forall x[(x + (-x) = 0) \wedge (x \cdot x^{-1} = 1)]$,

(iii) $\forall xy[(x + y = y + x) \wedge (x \cdot y = y \cdot x)]$,

(iv) $\forall xyz[(x + (y + z) = (x + y) + z) \wedge (x \cdot (y \cdot z) = (z \cdot y) \cdot z)]$,

(v) $\forall xyz[x \cdot (y + z) = x \cdot y + x \cdot z]$,

and every $\mathcal{L}$-sentence of the form

(vi) $\forall \bar{x} \exists y(f(\bar{x}, y) = 0)$

where $f(\bar{x}, y) \in \mathbb{Z}[\bar{x}, y]$. Then every algebraically closed field must satisfy all sentences in ACF. Conversely, if a mathematical structure satisfies all sentences in ACF, then the structure is always an algebraically closed field.

Remark

For any field K and $f(\bar{x}) \in K[\bar{x}]$, there exist $F(\bar{x}, \bar{y}) \in \mathbb{Z}[\bar{x}, \bar{y}]$ and $\bar{c} \in K$ such that $f(\bar{x}) = F(\bar{x}, \bar{c})$.

Definition

- Let $\varphi(x)$ is an $\mathcal{L}$-formula and M a mathematical structure. For $a \in M$, we say *a realize $\varphi(x)$ in M* and write $M \models \varphi(a)$ if $\varphi(a)$ is true in M. (e.g. $\mathbb{C} \models \varphi(i)$ where $\varphi(x) := \ulcorner x \cdot x + 1 = 0 \urcorner$.)
- Let T be a theory. We say a mathematical structure M is a *model of T* and write $M \models T$ if every sentence in T is true in M ($M \models \sigma$ for all $\sigma \in T$)
- For two structures M and N, we say *M is a substructure of N* if $M \subseteq N$.
- We say *M is an elementary substructure of N* and write $M \prec N$ if
  - M is a substructure of N,
  - $M \models \varphi(\bar{a})$ if and only if $N \models \varphi(\bar{a})$ for all $\varphi(x)$ and $\bar{a} \in M$.

Example

With respect to $\mathcal{L} = \{0, 1, +, -, \cdot, ^{-1}\}$, $\mathbb{R}$ is a substructure of $\mathbb{C}$ but $\mathbb{R} \not\prec \mathbb{C}$ since $\mathbb{R} \not\models \varphi(1)$ and $\mathbb{C} \models \varphi(1)$ where $\varphi(x) := \ulcorner \exists y(y \cdot y + x = 0) \urcorner$.

Definition

We say a theory T is *model complete* if $M \prec N$ for all $M, N \models T$ with $M \subseteq N$.

Fact

ACF is model complete.

Using this we can give an easy proof of Nullstellensatz.

Example (Nullstellensatz)

Let K be an algebraically closed field, I an ideal of $K[\bar{x}]$, and $g(\bar{x}) \in K[\bar{x}]$. If $g(\bar{x}) \notin \sqrt{I}$, then there exists $\bar{a} \in K$ such that $g(\bar{a}) \neq 0$ and $f(\bar{a}) = 0$ for all $f \in I$.

Proof. Choose any prime ideal $P \subseteq K[\bar{x}]$ with $I \subseteq P$ and $g \notin P$. Let L be the algebraic closure of the fraction field of $K[\bar{x}]/P$. Then $K, L \models ACF$ and $K \subseteq L$. Thus $K \prec L$. Note that there exist $f_0(\bar{x}), ...f_n(\bar{x}) \in K[\bar{x}]$ such that $\langle f_0, ..., f_n \rangle = I$. By the choice of L, there exists $\bar{b} \in L$ such that $g(\bar{b}) \neq 0$ and $f_i(\bar{b}) = 0$ for all $i \leq n$. Thus $L \models \varphi(0, \bar{d}, \bar{c}_0, ..., \bar{c}_n)$ where $\varphi(z, \bar{u}, \bar{v}_0, ...\bar{v}_n) := \ulcorner \exists \bar{x}(G(\bar{x}, \bar{u}) \neq z \wedge F_0(\bar{x}, \bar{v}_0) = z \wedge \cdots \wedge F_n(\bar{x}, \bar{v}_n) = z) \urcorner$ and $\bar{d}, \bar{c}_0, ..., \bar{c}_n \in K$, where $G(\bar{x}, \bar{y}), F_0(\bar{x}, \bar{y}), ..., F_n(\bar{x}, \bar{y}) \in \mathbb{Z}[\bar{x}, \bar{y}]$ and $G(\bar{x}, \bar{d}) = g(x), F_0(\bar{x}, \bar{c}_0) = f_0(\bar{x}), ..., F_n(\bar{x}, \bar{c}_n) = f_n(\bar{x})$. Since $K \prec L$, we have $K \models \varphi(0)$. Thus there exists $\bar{a} \in K$ we want.

For a given theory T, we may assume that there exists a sufficiently large model of T, say $\mathbb{M}$, such that every small model of T is an elementary substructure of $\mathbb{M}$. This $\mathbb{M}$ is called a *saturated model*. From now unless we mention otherwise, every set or model is in the saturated model $\mathbb{M}$.

Definition

Let $\mathcal{L}$ be a language, T an $\mathcal{L}$-theory, and $\mathbb{M}$ a saturated model of T.

- A set $X \subseteq \mathbb{M}$ is said to be *$\mathcal{L}$-definable* if there exists an $\mathcal{L}$-formula $\varphi(x)$ such that $a \in X$ if and only if $\mathbb{M} \models \varphi(a)$. In this case, we say *$\varphi$ defines X*. (e.g. the set of all solutions of $f(\bar{x}) = 0$ in a field K and $f(\bar{x}) \in \mathbb{Z}[\bar{x}]$)
- We say a set X is *$\mathcal{L}$-definable over b* if there exists an $\mathcal{L}$-formula $\varphi(x, y)$ such that $a \in X$ if and only if $\mathbb{M} \models \varphi(a, b)$. In this case, X is defined by $\varphi(x, b)$ and we call b a *parameter*. (e.g. the set of all solutions of $f(\bar{x}) = 0$ in a field K and $f(\bar{x}) \in K[\bar{x}]$)
- A class of sets $\{X_\alpha : \alpha < \kappa\}$ is said to be *uniformly definable* if there exists a $\varphi(x, y)$ and $\{a_\alpha : \alpha < \kappa\}$ such that $\varphi(x, a_\alpha)$ defines X_α for each $\alpha < \kappa$.

Note that if a language becomes richer, then there will be more definable sets.

Definition

Let $\mathcal{L}$ be a language containing a binary symbol $<$. Let T be an $\mathcal{L}$-theory containing sentences that say $<$ is an order relation. Then we say T is *o-minimal* if every $\mathcal{L}$-definable set X defined by a formula with single free variable, (in the saturated model $\mathbb{M}$) is a union of finitely many intervals.

Example

The following are o-minimal.

- $\mathcal{L} = \{<\}$, T is an $\mathcal{L}$-theory of dense linear order.
- $\mathcal{L} = \{0, 1, +, -, \cdot, ^{-1}, <\}$, T is an $\mathcal{L}$-theory of real closed field (ordered field axiom + "every positive element has a square root" + "every polynomial of odd degree has a root")
- $\mathcal{L}_{exp} = \{0, 1, +, -, \cdot, ^{-1}, <, exp\}$, R_{exp} is an $\mathcal{L}$-theory of real exponential field. (real closed field with the exponential function exp)

Fact [Wilkie, 1996]

- In $K \models R_{exp}$, for each formula $\varphi(\bar{x}) \in \mathcal{L}_{\exp}$, there exist finitely many $f_i(\bar{x}, \bar{y}, exp(\bar{x}), exp(\bar{y})) \in \mathbb{Z}[\bar{x}, \bar{y}, exp(\bar{x}), exp(\bar{y})]$ such that $\varphi(\bar{x})$ and $\exists \bar{y} \bigwedge (f_i(\bar{x}, \bar{y}, exp(\bar{x}), exp(\bar{y})) = 0)$ define the same set. Thus every definable set in K^m is the projection of some exponential variety in K^n. $(n \geq m)$
- The theory of real exponential fields is model complete.

Definition

- *ict-pattern of depth* κ is a class of definable sets $\{X_{\alpha,i} : \alpha < \kappa, i < \omega\}$ such that
  - $\{X_{\alpha,i} : i < \omega\}$ is uniformly definable for each $\alpha < \kappa$,
  - for each $\sigma : \kappa \to \omega$, there exists $a \in M$ such that $a \in X_{\alpha,\sigma(\alpha)} \setminus \bigcup_{j \neq \sigma(\alpha)} X_{\alpha,j}$ for all $\alpha < \kappa$.
- *dp-rank* of M is the supremum of cardinal κ such that there exists an ict-pattern of depth κ in M.
- We say a structure M is *dp-minimal* if its dp-rank is 0 or 1.

If a language becomes richer, then there will be more definable sets, and thus we have a larger dp-rank.

Model theoretic features can also be used to derive general properties of certain mathematical structures.

Example

- Algebraically closed fields are dp-minimal.
- Real closed fields are dp-minimal.
- Finite extensions of the p-adic numbers $\mathbb{Q}_p$ are dp-minimal.

Fact [Johnson, 2018]

If K is a dp-minimal field, then it is a perfect field and one of the following holds:

- K is finite,
- K is algebraically closed,
- K is real closed,
- K has a non-trivial henselian valuation.

As above, theories can be classified according to the possibility of the existence of a class of definable sets that satisfy certain combinatorial patterns.

Definition

Let T be a theory. If there exists a class of uniformly definable sets $\{X_\eta : \eta$ is a node in the binary tree $2^{<\omega}\}$ such that

- $X_{\eta_0}, ..., X_{\eta_n}$ have a common solution if $\eta_0, ..., \eta_n$ lie in the same path,
- X_η, X_ν have no common solution if η, ν do not lie in the same path,

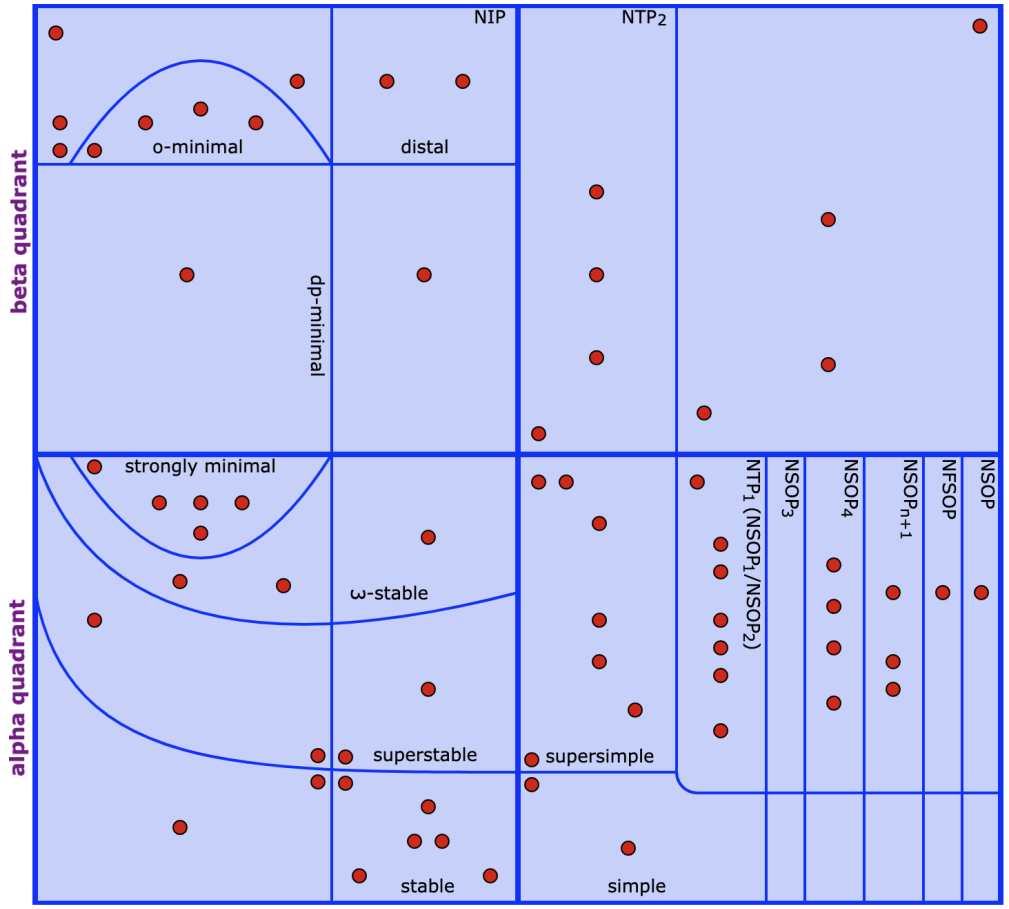
in any model M of T, then we say T is SOP_2. And if there is no such pattern, then T is said to be $NSOP_2$.

Definition [Ahn, K., 2020]

We say a theory T is ATP if there is a class of uniformly definable sets $\{X_\eta : \eta \in 2^{<\omega}\}$ such that

- $X_{\eta_0}, ..., X_{\eta_n}$ have a common solution if η_i, η_j do not lie in the same path for each $i < j \leq n$.
- X_η, X_ν have no common solution if η, ν lie in the same path.

If there is no such pattern, then T is said to be $NATP$.



Nice Properties of Theories

ω-stable	superstable		stable
strongly minimal	dp-minimal		o-minimal
supersimple	simple	NIP	distal
NTP_1 ($NSOP_1$/$NSOP_2$)		NTP_2	NSOP
$NSOP_3$	$NSOP_4$	$NSOP_{n+1}$	NFSOP

Click a property above to highlight region and display details. Or click the map for specific region information.

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List of Examples

Implications Between Properties

Open Regions

Open Examples

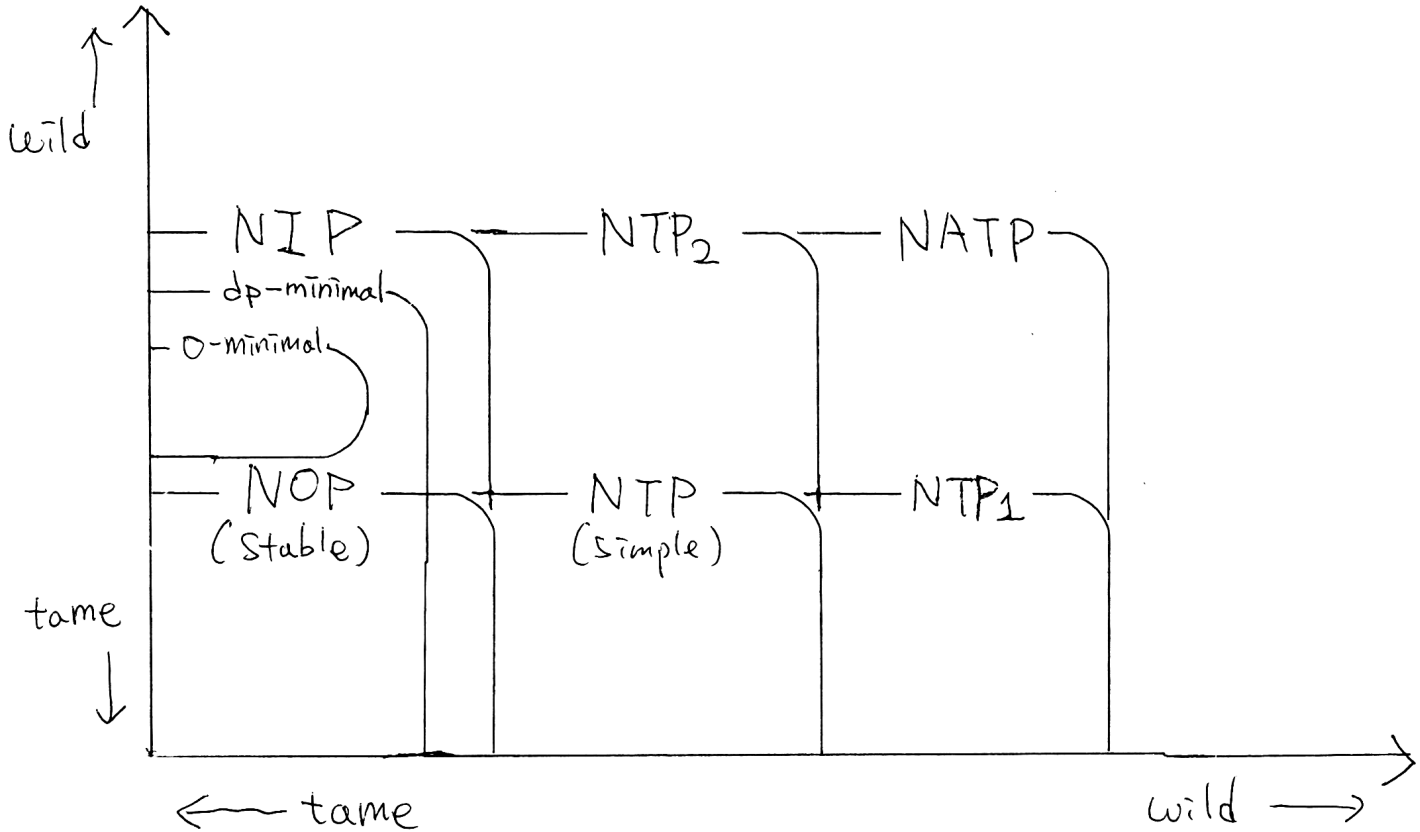
The following examples are theories whose exact positions on the map are unknown. Click an example to see its definition, and to highlight the possible regions it could be in.

- $(\mathbb{C}(t), +, \cdot, 0, 1)$
- Farey graph
- Artin braid groups

Questions? Suggestions? Corrections? email me: conant.38@osu.edu

References **Update Log**

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wild
NIP
NTP2
NATP
dp-minimal
0-minimal
NOP
(Stable)
NTP
(Simple)
NTP1
tame
tame
wild

Definition

A *pre-independence relation* $\downarrow$ is an invariant ternary relation on sets (i.e. $\downarrow \subseteq \mathcal{P}(\mathbb{M})^3$). If a triple of sets (A, B, C) is in the pre-independence relation $\downarrow$, then we write it $A \downarrow_C B$ and say *A is $\downarrow$-independent from B over C*.

- Monotonicity over C: If $AA' \downarrow_C BB'$, then $A \downarrow_C B$.
- Base monotonicity over C: If $A \downarrow_C BB'$, then $A \downarrow_{CB} B'$.
- Transitivity over D: If $A \downarrow_{DB} C$ and $B \downarrow_D C$, then $AB \downarrow_D C$.
- Right extension over C: If $A \downarrow_C B$, then for all D, there exists $D' \equiv_{CB} D$ (i.e. there exists an automorphism sending D' to D, fixing CB) such that $A \downarrow_C BD'$.
- Existence over C: $A \downarrow_C \emptyset$ for all A.
- Finite character over C: If $A \not\downarrow_C B$, then there exist finite $a \subseteq A$ and $b \subseteq B$ such that $a \not\downarrow_C b$.
- Symmetry over C: If $A \not\downarrow_C B$, then $B \not\downarrow_C A$.

Definition

- For a and b, $tp(a/b)$ is the set of all formulas $\varphi(x, b)$ realized by a.
- Let A be a set and $\Sigma(x)$ a set of formulas. We say $\Sigma(x)$ is *finitely satisfiable in A* if every finite subset $\Sigma_0(x)$ of $\Sigma(x)$ has a common solution in A.
- We say a formula $\varphi(x, a)$ is *algebraic* if it has only finitely many solutions.
- For a set A, $acl(A)$ is the set of all m such that there exists an algebraic formula $\varphi(x, a)$ with $a \in A$ and $\models \varphi(m, a)$.
- We write $A \mathop{\smile\!\!\!\!\!\!|}^{u}_{C} B$ if $\mathrm{tp}(A/BC)$ is finitely satisfiable in C.
- We write $A \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} B$ if $\mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C)$.

Fact

Let T be a theory. If M is a model of T, then

- $\mathop{\smile\!\!\!\!\!\!|}^{u}$ satisfies existence, monotonicity, base monitonicity, left extension, right extension, transitivity, and finite character over M.
- $\mathop{\smile\!\!\!\!\!\!|}^{a}$ satisfies existence, monotonicity, right extension, symmetry, and finite character over M.
- In a vector space V, $a \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} b$ if and only if a and b are linearly independence over C.

Pre-independence relation can provide a criterion for a theory on whether it has model theoretic tree property.

Fact [Kim, Pillay, 1997]

A theory T is NTP if and only if there exists a pre-independence relation $\downarrow$ satisfying monotonicity, right extension, symmetry, transitivity, finite character, local character, and independence theorem over models, where

- Local character: For any a, B there exists $A \subseteq B$ such that $|A| < |T|$ and $a \downarrow_A B$.
- Independence thoerem: If $A \downarrow_M B$, $a \downarrow_M A$, and $b \downarrow_M B$, then there exists c such that $c \equiv_{MA} a$, $c \equiv_{MB} b$, and $c \downarrow_M AB$.

Fact [Kaplan, Ramsey, 2017]

A theory T is NTP_1 if and only if there exists a pre-independence relation $\downarrow$ satisfying monotonicity, right extension, symmetry, strong finite character, and independence theorem over models.

Definition

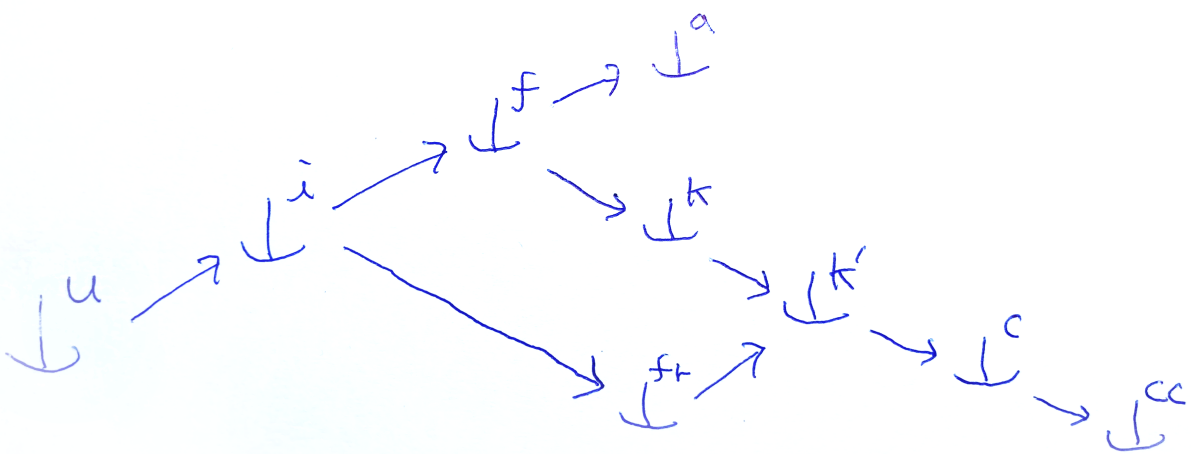
- We say a sequence $(a_i)_{i<\omega}$ is *indiscernible over a set A* if $a_{i_0}...a_{i_n} \equiv_A a_0...a_n$ for all $i_0 < \cdots < i_n$.
- We say a formula $\varphi(x, a)$ *divides over a set A* if there exists an indiscernible sequence $(a_i)_{i<\omega}$ with $a_0 = a$ such that $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent.
- We say a set of formula $\Sigma(x)$ *forks over a set A* if Σ implies a finite disjunction of dividing formulas over A ($\Sigma \vdash \bigvee \psi_i(x, a_i)$ and each ψ_i divides over A).
- For a pre-independence relation $\downarrow$, we say a sequence $(a_i)_{i<\omega}$ is *$\downarrow$-Morley sequence over a set A* if
  - $a_i \downarrow_A (a_j)_{j<i}$ for all $i < \omega$,
  - $(a_i)_{i<\omega}$ is indiscernible over A.
- We say *$\varphi(x, a)$ $\downarrow$-divides over A* if there exists a $\downarrow$-Morley sequence $(a_i)_{i<\omega}$ with $a_0 = a$ and $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent.
- We say a set of formula $\Sigma(x)$ *$\downarrow$-forks over a set A* if Σ implies a finite disjunction of $\downarrow$-dividing formulas over A.

Note that forking is $\downarrow$-forking where $\downarrow = \mathbb{M}^3$.

Definition/Fact

- We write $A \downarrow^f_C B$ if $\mathrm{tp}(A/BC)$ has no forking formula over C.
- We write $A \downarrow^K_C B$ if $\mathrm{tp}(A/BC)$ has no $\downarrow^f$-forking formula over C.
- $A \downarrow^f_C B$ implies $A \downarrow^K_C B$.

It is known that $\downarrow^f$ and $\downarrow^K$ satisfy Kim and Pillay's criterion, Kaplan and Ramsey's one above, respectively.



Using notions of forking and dividing, we can give a rank on type-definable sets in $\mathbb{M}$.

Definition (SU-rank)

We define $SU(\text{tp}(a/A)) \leq \alpha + 1$ if for some $B \supseteq A$, there exists $a' \equiv_A a$ such that $a' \not\downarrow^f_A B$ and $SU(\text{tp}(a'/B)) \leq \alpha$.

Fact

Suppose that T is NTP.

- $SU(\text{tp}(a/B)) = SU(\text{tp}(a/A))$ whenever $a \downarrow^f_A B$.
- $SU(\text{tp}(a/bA)) + SU(\text{tp}(b/A)) \leq SU(\text{tp}(ab/A)) \leq SU(\text{tp}(a/bA)) + SU(\text{tp}(b/A))$.
- If $a \downarrow^f_A b$, then $SU(\text{tp}(ab/A)) = SU(\text{tp}(a/A)) + SU(\text{tp}(b/A))$.
- If $SU(\text{tp}(a/bA)) < \infty$ and $SU(\text{tp}(a/A)) \geq SU(\text{tp}(a/Ab)) + \alpha$, then $SU(\text{tp}(b/A)) \geq SU(\text{tp}(b/Aa)) + \alpha$.

SU-rank can be useful for model-theoretic analysis of groups, when the group is an NTP with respect to an expanded language of the group language $\mathcal{L} = \{+, -, 0\}$.

Similarly, one can expect that $\downarrow^K$ plays an important role giving an appropriate rank notion on NTP_1 theories. But it has not yet been developed since there is still a lot we don't know about $\downarrow^K$.

Theorem [Kim, K., Lee, 2023]

In NTP_1, $\downarrow^K$ satisfies existence over an arbitrary set.

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