Kim-dividing in NATP theories

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In model theory, theories can be classified according to the possibility of the existence of a class of definable sets that satisfy certain combinatorial patterns.

Definition

Let L be a language, T an L-theory, $\mathbb{M}$ a monster model of T.
T has TP if there exist an L-formula $\varphi(x,y)$, $k<\omega$, and a tree-indexed set of tuples of parameters $(a_\eta)_{\eta\in\omega^{<\omega}}$ in $\mathbb{M}$ such that

- $\{\varphi(x,a_{\eta_0}),...,\varphi(x,a_{\eta_n})\}$ is consistent if $\eta_0 \trianglelefteq \cdots \trianglelefteq \eta_n$,
- $\{\varphi(x,a_{\eta^\frown\langle i_0\rangle}),...,\varphi(x,a_{\eta^\frown\langle i_k\rangle})\}$ is inconsistent for all $\eta\in\omega^{<\omega}$ and $i_0<\cdots<i_k<\omega$.

T has TP_1 if there exist an L-formula $\varphi(x,y)$, $k<\omega$, and a tree-indexed set of tuples of parameters $(a_\eta)_{\eta\in\omega^{<\omega}}$ in $\mathbb{M}$ such that

- $\{\varphi(x,a_{\eta_0}),...,\varphi(x,a_{\eta_n})\}$ is consistent if $\eta_0 \trianglelefteq \cdots \trianglelefteq \eta_n$,
- $\{\varphi(x,a_{\eta_0}),...,\varphi(x,a_{\eta_k})\}$ is inconsistent if $\eta_i \not\trianglelefteq \eta_j$ and $\eta_j \not\trianglelefteq \eta_i$ $(\eta_i \perp \eta_j)$ for all $i<j\leq k$.

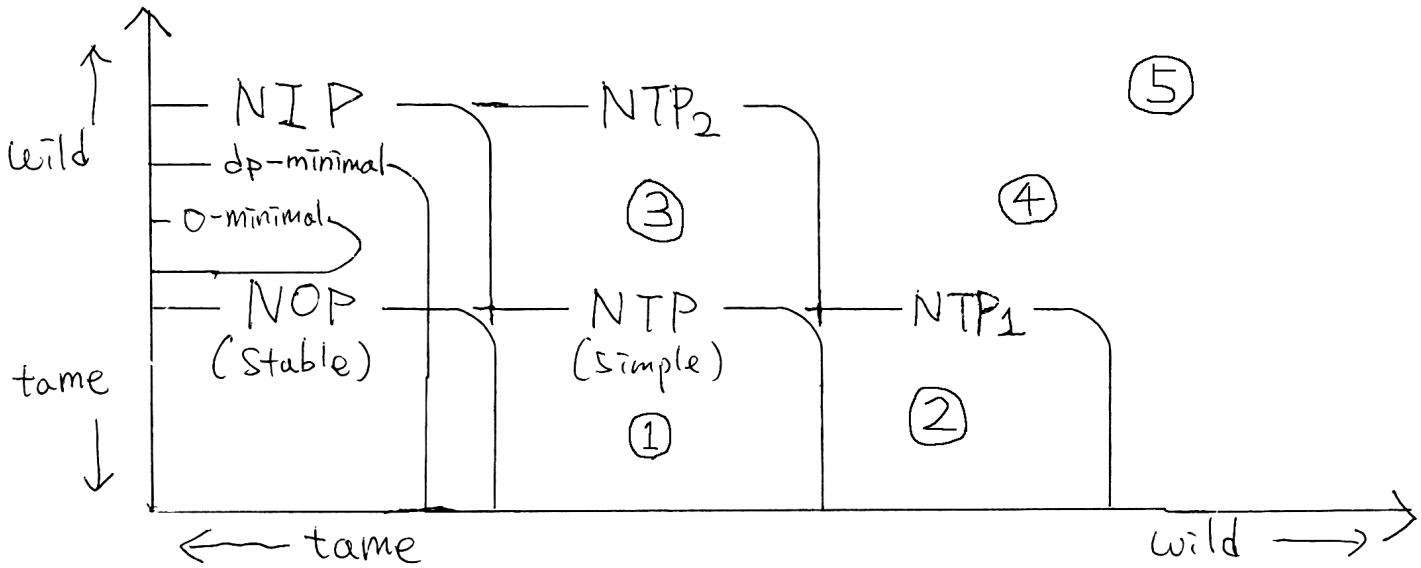
T has TP_2 if there exist an L-formula $\varphi(x,y)$ and an array-indexed set of tuples of parameters $(a_\eta)_{\eta\in\omega\times\omega}$ in $\mathbb{M}$ such that

- $\{\varphi(x,a_{i,f(i)}) : i<\omega\}$ is consistent for all $f:\omega\to\omega$,
- $\{\varphi(x,a_{i,j}),\varphi(x,a_{i,k})\}$ is inconsistent for all $i,j,k<\omega$ with $j\neq k$.

T is NTP (NTP_1, NTP_2) if it does not have TP (TP_1, TP_2 respectively).

Fact [Shelah]

T has TP if and only if it has TP_1 or TP_2.



Examples

1. Random graph in the language $\{R : \text{a binary relation symbol}\}$ is NTP.
2. [Kaplan, Ramsey] ω-free pseudo algebraically closed fields with characteristic 0 in $\{0, 1, +, -, \times\}$ are NTP_1 and has TP.
3. [Montenegro] Bounded pseudo real closed fields in $\{0, 1, +, -, \times\}$ are NTP_2 and has TP.
4. [Tran] Algebraically closed field with circular order (ACFO) in $\{0, 1, +, -, \times, \square^{-1}, \lhd\}$ has TP_1 and TP_2.
5. Atomless Boolean algebra (ABA) in $\{0, 1, \neg, \wedge, \vee\}$ has TP_1 and TP_2.

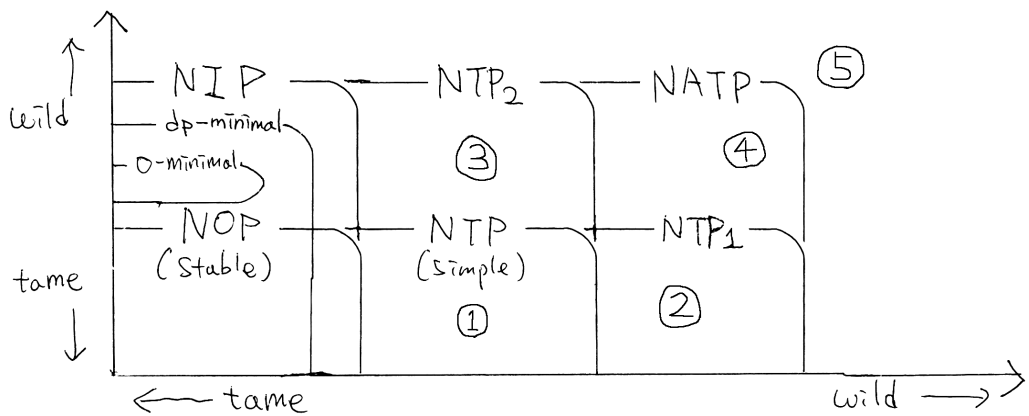
Definition [Ahn, K.]

- $X \subseteq \omega^{<\omega}$ is an *antichain* if $\eta \perp \nu$ for all $\eta, \nu \in X$.
- T has *ATP (antichain tree property)* if there exist an L-formula $\varphi(x, y)$, $k < \omega$, and $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  - $\{\varphi(x, a_{\eta_0}), ..., \varphi(x, a_{\eta_n})\}$ is consistent if $\{\eta_0, \cdots, \eta_n\}$ is an antichain,
  - $\{\varphi(x, a_{\eta_0}), ..., \varphi(x, a_{\eta_k})\}$ is inconsistent if $\{\eta_0, \cdots, \eta_k\}$ is a path.

  T is *NATP* if it does not have ATP.

Fact [Ahn, K.]

If T has ATP, then it has TP_1 and TP_2.



Example

④ [Ahn, K., Lee, Lee] ACFO is NATP.

⑤ [Ahn, K., Lee] ABA has ATP.

Definition

- A global type $p(x)$ is *invariant* over a set A if $\varphi(x,a) \leftrightarrow \varphi(x,a') \in p(x)$ for all $\varphi(x,y)$ and $a \equiv_A a'$. $p(x)$ is a *coheir* over A if $\varphi(x,a)$ is satisfiable in A for all $\varphi(x,a) \in p(x)$.
- A sequence $(a_i)_{i<\omega}$ is *indiscernible* over A if $a_{i_0}...a_{i_n} \equiv_A a_0...a_n$ for $i_0 < \cdots < i_n$.
- A sequence $(a_i)_{i<\omega}$ is an *invariant Morley sequence* (*coheir Morley sequence*) over A if there exists an invariant type (coheir) $p(x)$ over A such that $a_i \models p|_{Aa_{<i}}$ for all $i < \omega$.
- A formula $\varphi(x,a)$ *divides* (*Kim-divides, coheir-divides*) over a set A if there exists an indiscernible sequence (invariant Morley sequence, coheir Morley sequence) $(a_i)_{i<\omega}$ with $a_0 = a$ such that $\{\varphi(x,a_i)\}_{i<\omega}$ is inconsistent.
- A formula $\varphi(x,a)$ *forks* (*Kim-forks, coheir-forks*) over a set A if there are dividing (Kim-dividing, coheir-dividing) formulas $\psi_0(x,a_0),...,\psi_{n-1}(x,a_{n-1})$ over A such that $\varphi(x,a) \vdash \bigvee \psi_i(x,a_i)$.
- A sequence $(a_i)_{i<\omega}$ is a *Morley sequence* over a set A if it is indiscernible over A and $\mathrm{tp}(a_i/Aa_{<i})$ has no forking formula over A for all $i < \omega$.

Fact

- If a sequence is coheir Morley, then it is invariant Morley.
- If a sequence is invariant Morley, then it is Morley.
- coheir-dividing $\Rightarrow$ Kim-dividing $\Rightarrow$ dividing

Fact

- [Kim] In NTP theories, if $\varphi(x, a)$ divides over a set A, then it coheir-divides over A.
- [Kaplan, Ramsey] In NTP_1 theories, if $\varphi(x, a)$ Kim-divides over a model M, then it coheir-divides over M.
- [Chernikov, Kaplan] In NTP_2 theories, if $\varphi(x, a)$ divides over a model M, then it coheir-divides over M.

In fact, stronger results hold in NTP and NTP_1 theories. Namely,

Kim's lemma

- [Kim] In NTP, if $\varphi(x, a)$ divides over a set A, then $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent for any Morley sequence $(a_i)_{i<\omega}$ over A with $a_0 = a$.
- [Kaplan, Ramsey] In NTP_1, if $\varphi(x, a)$ Kim-divides over a model M, then $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent for any invariant Morley sequence $(a_i)_{i<\omega}$ over M with $a_0 = a$.

From Kim's lemma, we can derive the equivalent of forking and dividing in NTP and the equivalent of Kim-forking and Kim-dividing in NTP_1.

Fact

- [Kim] In NTP, for any set A and tuple a, there exists a Morley sequence $(a_i)_{i<\omega}$ over A with $a_0 = a$.
- In any theory, for any model M and tuple a, there exists an invariant Morley sequence $(a_i)_{i<\omega}$ over M with $a_0 = a$.

Suppose that $\varphi(x, a)$ forks over A in an NTP theory. Then there exist dividing formulas $\psi_0(x, a_0), ..., \psi_{n-1}(x, a_{n-1})$ over A such that $\varphi(x, a) \vdash \bigvee \psi_i(x, a_i)$. By the fact above, we can choose a Morley sequence $(\bar{b}_i := (b_0^i, ..., b_n^i))_{i<\omega}$ over A with $\bar{b}_0 = (a_0, ..., a_{n-1}, a)$. If $\{\varphi(x, b_n^i)\}_{i<\omega}$ is consistent, then by pigeon hole principle, there is $k < n$ such that $\{\psi_k(x, b_k^i)\}_{i<\omega}$ is consistent. But by Kim's lemma, it is a contradiction since $(b_k^i)_{i<\omega}$ is a Morley sequence. The same argument works for NTP_1.

In NTP_2 cases, we need a little more complicated approach to show the equivalent between forking and dividing.

Definition

- A global type $p(x)$ is *strictly invariant* over a set A if it is invariant over A and $\mathrm{tp}(b/Aa)$ has no forking formula over A when $a \models p|_{Ab}$. A sequence $(a_i)_{i<\omega}$ is a *strictly invariant Morley sequence* over A if there is a strictly invariant type $p(x)$ over A such that $a_i \models p|_{Aa_{<i}}$ for all $i < \omega$.
- $\varphi(x, a)$ *quasi-divides* over A if $\{\varphi(x, a_0), ..., \varphi(x, a_n)\}$ is inconsistent for some $a_0, ..., a_n \equiv Aa$.

[Chernikov, Kaplan] Strategy of proving forking=dividing in NTP_2

Assume T is NTP_2 and $M \models T$ is a model. Chernikov and Kaplan proved the equivalence of Kim-forking and Kim-dividing as follows.

(1) If a formula divides over M, then it coheir-divides over M.

(2) If a formula forks over M, then it quasi-divides over M by Bloom Lemma and (1).

(3) Using (2), one can show that there is a strictly invariant Morley sequence starting with b over M for all b and a model M.

(4) (Kim's Lemma in NTP_2) If $\varphi(x, b)$ divides over M, then $\{\varphi(x, b_i)\}_{i\in\omega}$ is inconsistent for any strictly invariant Morley sequence $(b_i)_{i\in\omega}$ with $b_0 = b$ over M.

(5) By (3) and (4), if a formula forks over M, then it Kim-divides over M. Thus it divides over M.

- In NTP, forking and dividing are equivalent over sets.
- In NTP_1, Kim-forking and Kim-dividing are equivalent over models.
- In NTP_2, forking and dividing are equivalent over models.

The goal of our work, which is still ongoing, is to show that Kim-forking and Kim-dividing are equivalent in NATP theories. For this we are trying to apply the strategy of Chernikov and Kaplan on NATP theories, and have obtained similar observations to the previous few steps.

Main Theorem [K., Lee]

In NATP theories, (1) if $\varphi(x, a)$ Kim-divides over a model M, then it coheir-divides over M.

sketch of proof

The key ingredients of the proof are left extension of $\downarrow^u$, the fact that the class of coheir is closed under taking a tensor power, and that the number of coheirs is bounded.
Suppose $\varphi(x, a)$ Kim-divides over M. Then there exists a global invariant type $p(x)$ over M containing $\mathrm{tp}(a/M)$ such that $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent for all $(a_i)_{i<\omega}$ with $a_i \models p|_{Ma_{<i}}$ for each $i < \omega$. Suppose that $\varphi(x, a)$ does not coheir-divide over M. Choose any coheir $q_0(x)$ over M containing $\mathrm{tp}(a/M)$. Let $aa' \models (q_0(x) \otimes q_0(y))|_M$ and $a'' \models p|_{Maa'}$. Then there exists a coheir $q_1(x, y, z)$ containing $q_0(x) \otimes q_0(y) \cup \mathrm{tp}_{xyz}(aa'a''/M)$. By repeating this, we can find a coheir $q((x_\eta)_{\eta \in \mathcal{T}_\omega})$ such that if $(a_\eta)_{\eta \in \mathcal{T}_\omega} \models q|_M$, then

- $\{a_{\eta_0}, ..., a_{\eta_n}\}$ is a finite coheir Morley sequence with $a_{\eta_0} \equiv_A a$ for each antichain $\eta_0 <_{lex} \cdots <_{lex} \eta_n$,
- $\{a_{\eta_0}, ..., a_{\eta_n}\}$ is a finite invariant Morley sequence generated by p over M for each $\eta_0 \lhd \cdots \lhd \eta_n$.

Thus $(a_\eta)_{\eta \in \mathcal{T}_\omega} \models q|_M$ forms an antichain tree with $\varphi(x, a)$. But we assume T is NATP, it is a contradiction.

Hanson found an example of a theory having a Kim-dividing formula over some model which does not coheir-divides over the same model.

Definition

A global type $p(x)$ is *Kim-strictly invariant* over a set A if it is invariant over A and $\mathrm{tp}(b/Aa)$ has no Kim-forking formula over A when $a \models p|_{Ab}$. A sequence $(a_i)_{i<\omega}$ is a *Kim-strictly invariant Morley sequence* over A if there is a Kim-strictly invariant type $p(x)$ over A such that $a_i \models p|_{Aa_{<i}}$ for all $i < \omega$.

Corollary [K., Lee]

In NATP theories,

(2) if $\varphi(x, a)$ Kim-forks over a model M, then it quasi-divides over M by Broom lemma and (1),

(3) there exists a Kim-strictly invariant Morley sequence starting with b over M for all b and a model M by (2).

Recently, Kruckman and Ramsey proved a stronger result. They proved that in fact Kim-forking always implies quasi-dividing over an arbitrary set, in any theory. So the existence of Kim-strictly invariant Morley sequences always holds for any tuple and any base set, in any theory.

Let X_M be a class of invariant Morley sequences over a model M such that $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent for any $(a_i)_{i<\omega} \in X_M$ and Kim-dividing formula $\varphi(x, a)$ over M.

Proposition [K., Lee]

If T is NATP and $M \models T$, then X_M is a subclass of the class of Kim-strictly invariant Morley sequences over M.

Proposition [K., Lee]

Let T be NATP and $M \models T$. If there exists a pre-independence relation $\downarrow$ stronger than $\downarrow^h$ and satisfies monotonicity, full existence, right chain condition for coheir Morley sequences, and strong right transitivity over M, then the class of invariant Morley sequences generated by $\downarrow$ is a subclass of X_M.

NATP, NBTP, and NCTP

Definition [Kruckman, Ramsey]

- A *left-leaning path* is a finite sequence $(\eta_0, ..., \eta_n)$ such that if $\eta_k = \nu^\frown\langle i\rangle$, then $\nu^\frown\langle j\rangle \lhd \eta_{k+1}$ for some $j \leq i$.
- A *right-veering path* is a finite sequence $(\eta_0, ..., \eta_n)$ such that if $\eta_k = \nu^\frown\langle i\rangle$, then $\nu^\frown\langle j\rangle \trianglelefteq \eta_{k+1}$ for some $i < j$.
- T has *BTP* (*bizarre tree property*) if there are $k < \omega$, $\varphi(x, y)$, $(a_\eta)_{\eta\in\omega^{<\omega}\setminus\{\emptyset\}}$ such that
  - $\{\varphi(x, a_{\eta_0}), ..., \varphi(x, a_{\eta_n})\}$ is consistent for all left-leaning path $\eta_0, ..., \eta_n$.
  - $\{\varphi(x, a_{\eta_0}), ..., \varphi(x, a_{\eta_k})\}$ is inconsistent for all right-veering path $\eta_0, ..., \eta_k$.

  T is *NBTP* if it does not have BTP.

Definition [Mutchnik]

T has *CTP* (*comb tree property*) if there are $k < \omega$, $\varphi(x, y)$, $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that

- $\{\varphi(x, a_{\eta_0}), ..., \varphi(x, a_{\eta_n})\}$ is consistent for any antichain $\eta_0 <_{lex} \cdots <_{lex} \eta_n$ with $\eta_0 \wedge \eta_k \not\trianglelefteq \eta_{k+1}$ for all $0 \leq k < n$.
- $\{\varphi(x, a_{\eta_0}), ..., \varphi(x, a_{\eta_k})\}$ is inconsistent for all $\eta_0 \trianglelefteq \cdots \trianglelefteq \eta_k$.

T is *NCTP* if it does not have CTP.

Fact [K., Lee]

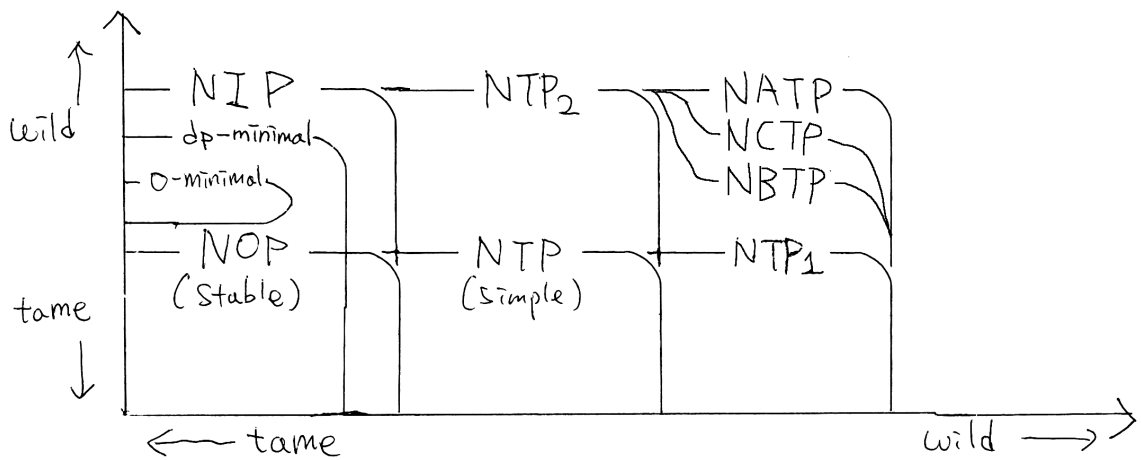
If T has ATP, then it has CTP.

Fact [Hanson], [Kruckman, Ramsey]

If T has CTP, then it has BTP.

Fact [Kruckman, Ramsey]

If T has BTP, then it has TP_1 and TP_2.



Kim's lemma in NBTP [Kruckman, Ramsey]

In NBTP theories, if $\varphi(x, a)$ Kim-divides over a model M, then $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent for any Kim-strictly invariant Morley sequence $(a_i)_{i<\omega}$ over M.

Kim's lemma in NCTP (1) [Mutchnik]

In NCTP theories, if $\varphi(x, a)$ coheir-divides over a model M, then $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent for any canonical Morley sequence $(a_i)_{i<\omega}$ over M.

Kim's lemma in NCTP (2) [Hanson]

The following are equivalent.

- T is NCTP.
- For all $\varphi(x, a)$ and a model M, if $\varphi(x, a)$ coheir-divides over M, then $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent for any heir-coheir Morley sequence $(a_i)_{i<\omega}$ over M with $a_0 = a$.

Question

In NATP, is there a class of invariant Morley sequences, say X, satisfying the following statement?

If $\varphi(x, a)$ Kim-divides over a model M, then $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent for any invariant Morley sequence $(a_i)_{i<\omega} \in X$.

Thank you

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