Existence axiom of pre-independence relaions in NSOP_1 theories

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This talk is based on an ongoing work with Byunghan Kim and Hyoyoon Lee.

- Finding a well-behaved notion of independence in NSOP_1
- Kaplan and Ramsey have already introduced $\downarrow^K$ and showed that it works very well in NSOP_1 theories, over models.
- Later, Dobrowolski, Kim, and Ramsey showed that if we assume existence axiom of $\downarrow^f$, then $\downarrow^K$ works over an arbitrary set as well (in NSOP_1 theories).

We want to remove the 'existence axiom', and there are two (or more) ways we can try to do that.

1. Showing that $\downarrow^f$ always satisfies existence over an arbitrary set in NSOP_1 theories.
2. Finding a pre-independence relation satisfying existence that can play the same roles as $\downarrow^f$.

While trying the second strategy, we recently proved that in NSOP_1 theories, if a pre-independence relation satisfies certain conditions, then it induces another pre-independence relation satisfying existence over an arbitrary set.

There exists an example of SOP_1 theory which is not satisfying existence axiom of $\downarrow^f$. We still do not know whether being NSOP_1 implies satisfying existence axiom of $\downarrow^f$.

Definition

A *pre-independence relation* $\downarrow$ is an invariant ternary relation on sets. If $(A, B, C) \in \downarrow$, then we write $A \downarrow_C B$ and say *A is $\downarrow$-independent from B over C*.

- Monotonicity over C: If $AA' \downarrow_C BB'$, then $A \downarrow_C B$.
- Base monotonicity over C: If $A \downarrow_C BB'$, then $A \downarrow_{CB} B'$.
- Transitivity over D: If $A \downarrow_{DB} C$ and $B \downarrow_D C$, then $AB \downarrow_D C$.
- Right extension over C: If $A \downarrow_C B$, then for all D, there exists $D' \equiv_{CB} D$ such that $A \downarrow_C BD'$.
- Existence over C: $A \downarrow_C \emptyset$ for all A.
- Symmetry over C: If $A \downarrow_C B$, then $B \downarrow_C A$.
- Finite character over C: If $A \not\downarrow_C B$, then there exist finite $a \subseteq A$ and $b \subseteq B$ such that $a \not\downarrow_C b$.
- Strong finite character: If $A \not\downarrow_C B$, then there is a finite $b \subseteq B$ and $\varphi(x, y)$ such that $\varphi(x, b) \in \mathrm{tp}(A/BC)$ and $a \not\downarrow_C b$ for all $a \models \varphi(x, b)$.
- Local character over C: For A and B, there exists $C \subseteq B$ such that $|C| \leq |T|$ and $A \downarrow_C B$
- Independence theorem over C: If $A \downarrow_C B$, $A' \downarrow_C B'$, $B \downarrow_C B'$, and $A \equiv^L_C A'$, then there is A'' such that $A'' \equiv^L_{CB} A$, $A'' \equiv^L_{CB'} A'$, and $A'' \downarrow_C BB'$.
  (recall that $a \equiv^L_D a'$ if there are $a_0 = a, a_1, \ldots, a_k = a'$ such that a_i, a_{i+1} start an indiscernible sequence over D for each i.)

Example

- $\downarrow^d$: $a \downarrow^d_C b$ iff $\mathrm{tp}(a/Cb)$ does not divide over C.
- $\downarrow^f$: $a \downarrow^f_C b$ iff $\mathrm{tp}(a/Cb)$ does not fork over C.
- $\downarrow^a$: $a \downarrow^a_C b$ iff $\mathrm{acl}(aC) \cap \mathrm{acl}(bC) = \mathrm{acl}(C)$.
- $\downarrow^u$: $a \downarrow^u_C b$ iff $\mathrm{tp}(a/Cb)$ is finitely satisfiable in C. ($\mathrm{tp}(a/Cb)$ is a coheir extension of $\mathrm{tp}(a/C)$.)
- $\downarrow^h$: $a \downarrow^h_C b$ iff $\mathrm{tp}(a/Cb)$ is an heir extension of $\mathrm{tp}(a/C)$.
- $\downarrow^i$: $a \downarrow^i_C b$ iff there exists a global invariant type over C containing $\mathrm{tp}(a/Cb)$.

Fact

In general,

- $\downarrow^u \Rightarrow \downarrow^i \Rightarrow \downarrow^f \Rightarrow \downarrow^d \Rightarrow \downarrow^a$,
- $a \downarrow^u_C b \Leftrightarrow b \downarrow^h_C a$.

Non-forking independence $\downarrow^f$ gives an useful criteria for determining simple theories.

Fact [Kim, Pillay, 1997]

A theory T is simple if and only if there exists a pre-independence relation $\downarrow$ satisfying existence, right extension, symmetry, monotonicity, base monotonicity, transitivity, finite character, local character over sets, and the independence theorem over models. Moreover, if such $\downarrow$ exists, then $\downarrow = \downarrow^f$.

Kaplan and Ramsey generalize this to the case of NSOP_1.

Definition

Let L be a language, T an L-theory, $\mathbb{M}$ a monster model of T.
T has *TP* if there exist an L-formula $\varphi(x,y)$, $k<\omega$, and a tree-indexed set of tuples of parameters $(a_\eta)_{\eta\in\omega^{<\omega}}$ in $\mathbb{M}$ such that

- $\{\varphi(x,a_{\eta_0}),...,\varphi(x,a_{\eta_n})\}$ is consistent if $\eta_0 \trianglelefteq \cdots \trianglelefteq \eta_n$,
- $\{\varphi(x,a_{\eta^\frown\langle i_0\rangle}),...,\varphi(x,a_{\eta^\frown\langle i_k\rangle})\}$ is inconsistent for all $\eta\in\omega^{<\omega}$ and $i_0<\cdots<i_k<\omega$.

T has *SOP*$_1$ if there exist an L-formula $\varphi(x,y)$, $k<\omega$, and a tree-indexed set of tuples of parameters $(a_\eta)_{\eta\in\omega^{<\omega}}$ in $\mathbb{M}$ such that

- $\{\varphi(x,a_{\eta_0}),...,\varphi(x,a_{\eta_n})\}$ is consistent if $\eta_0 \trianglelefteq \cdots \trianglelefteq \eta_n$,
- $\{\varphi(x,a_{\eta^\frown\langle 1\rangle}),\varphi(x,a_\nu)\}$ is inconsistent if $\eta^\frown\langle 0\rangle \trianglelefteq \nu$.

T is *NTP* (*NSOP*$_1$) if it does not have TP (SOP$_1$, respectively).

Fact

- T is simple iff T is NTP.
- If T is NTP, then it is NSOP$_1$. The converse does not hold in general.

Kaplan and Ramsey generalize Kim-Pillay theorem to the case of NSOP$_1$.
To do that they first introduce the notion of Kim-independence $\mathop{\smile\!\!\!\!\!\!|}^{K}$.

Definition

- For a pre-independence relation $\downarrow$, we say a sequence $(a_i)_{i<\omega}$ is *$\downarrow$-Morley sequence over a set A* if
  - $a_i \downarrow_A (a_j)_{j<i}$ for all $i < \omega$,
  - $(a_i)_{i<\omega}$ is indiscernible over A.
- We say $\varphi(x, a)$ *$\downarrow$-Kim-divides over A* if there exists a $\downarrow$-Morley sequence $(a_i)_{i<\omega}$ over A with $a_0 = a$ and $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent.
- We say a set of formula $\Sigma(x)$ *$\downarrow$-Kim-forks over a set A* if Σ implies a finite disjunction of $\downarrow$-dividing formulas over A.
- We write $a \downarrow^{K^\downarrow}_C b$ if $\mathrm{tp}(a/Cb)$ does not $\downarrow$-Kim-fork over C.
- From now, we will write $\downarrow^{K^{\downarrow^i}}$, $\downarrow^{K^{\downarrow^d}}$, $\downarrow^{K^{\downarrow^f}}$, $\downarrow^{K^{\downarrow^a}}$, $\downarrow^{K^{\downarrow^u}}$, $\downarrow^{K^{\downarrow^h}}$ as $\downarrow^{K^i}$, $\downarrow^{K^d}$, $\downarrow^{K^f}$, $\downarrow^{K^a}$, $\downarrow^{K^u}$, $\downarrow^{K^h}$ for short, respectively.

Remark

- $\downarrow^f$ is $\downarrow^{K^\downarrow}$ where $\downarrow$ is the trivial pre-independence relation. (i.e. $a \downarrow_C b$ for all a, b, C)
- The original definition of Kim-independence $\downarrow^K$, introduced by Kaplan and Ramsey, is the same with $\downarrow^{K^i}$.
- $\downarrow^f \Rightarrow \downarrow^{K^{\downarrow'}} \Rightarrow \downarrow^{K^\downarrow}$ whenever $\downarrow \Rightarrow \downarrow'$. In particular, $\downarrow^f \Rightarrow \downarrow^{K^f} \Rightarrow \downarrow^{K^i}$.

Fact [Kim, Pillay, 1997]

A theory T is simple if and only if there exists a pre-independence relation $\downarrow$ satisfying monotonicity, existence, right extension, symmetry, base monotonicity, transitivity, finite character, and local character over sets, and the independence theorem over models. Moreover, if such $\downarrow$ exists, then $\downarrow = \downarrow^f$.

Fact [Kaplan, Ramsey, 2017]

A theory T is NSOP_1 if and only if there exists a pre-independence relation $\downarrow$ satisfying monotonicity, existence, right extension, symmetry, strong finite character, and independence theorem over models. Moreover, if such $\downarrow$ exists, then $\downarrow \Rightarrow \downarrow^K$ over models.

Remark

In the Kim-Pillay style criterion for NSOP_1 theories above, key properties are symmetry and independence theorem. Properties other than them and existence are relatively easy to obtain. For example, $\downarrow^{K^{\downarrow}}$ always satisfies monotonicity, right extension, and strong finite character over an arbitrary set in any theory, no matter how we choose $\downarrow$. Existence is not so trivial, but it is often not difficult to derive when the base set is a model.

Fact [Kaplan, Ramsey, 2017]

If T is NSOP_1, then $\mathop{\smile\!\!\!\!\!\!|}^{K^i}$ satisfies monotonicity, existence, right extension, symmetry, strong finite character, and independence theorem over models.

Question

In NSOP_1 theories, does $\mathop{\smile\!\!\!\!\!\!|}^{K^i}$ satisfy symmetry, or independence theorem over an arbitrary set?

$\mathop{\smile\!\!\!\!\!\!|}^{K^i}$ and $\mathop{\smile\!\!\!\!\!\!|}^{K^{\perp}}$coincide over models if $\mathop{\smile\!\!\!\!\!\!|}^{i} \Rightarrow \mathop{\smile\!\!\!\!\!\!|}$. In particular, $\mathop{\smile\!\!\!\!\!\!|}^{K^i}$and $\mathop{\smile\!\!\!\!\!\!|}^{K^f}$coincide.

Fact [Kaplan, Ramsey, 2017]

If T:NSOP_1 and $M \models T$, then $\varphi(x, a)$ $\mathop{\smile\!\!\!\!\!\!|}^{i}$-divides over M iff it $\mathop{\smile\!\!\!\!\!\!|}^{K^i}$-divides over M.

So we can rewrite the fact and question above, as below.

Fact [Kaplan, Ramsey, 2017]

If T is NSOP_1, then $\mathop{\smile\!\!\!\!\!\!|}^{K^f}$ satisfies monotonicity, existence, right extension, symmetry, strong finite character, and independence theorem over models.

Question

In NSOP_1 theories, does $\mathop{\smile\!\!\!\!\!\!|}^{K^f}$ satisfy symmetry, or independence theorem over an arbitrary set?

Fact [Dobrowolski, Kim, Ramsey, 2020]

In NSOP_1 theories, *if we assume that* $\downarrow^f$ *satisfies existence over sets*, then $\downarrow^{K^f}$ satisfies symmetry and independence theorem over sets.

Corollary [Dobrowolski, Kim, Ramsey, 2020]

Suppose T is NSOP_1. *If we assume that* $\downarrow^f$ *satisfies existence over sets*, then T is G-compact. i.e., $a \equiv_C^L b$ iff $a \equiv_C^{KP} b$.

($a \equiv_C^L b$ if there are $a_0 = a, a_1, \ldots, a_k = b$ such that a_i, a_{i+1} start an indiscernible sequence over C for each i. $a \equiv_C^{KP} b$ if $\models E(a, b)$ for any type definable equivalent relation $E(x, y)$ over D.)

- Can we remove '*If we assume that* $\downarrow^f$ *satisfies existence over sets*' in the above results?
- Or, can we replace $\downarrow^f$ in the assumption with another pre-independence relation to obtain similar results? That is, can we find $\downarrow$ whose induced pre-independence relation $\downarrow^{K^{\downarrow}}$ satisfies symmetry and independence theorem over sets?

Can we find $\downarrow$ whose induced pre-independence relation $\downarrow^{K^{\downarrow}}$ satisfies symmetry and independence theorem over sets?

If such $\downarrow$ exists, it is more likely to be found among those that are *weaker than* $\downarrow^f$, *stronger than* $\downarrow^{K^f}$, and satisfy *existence*. It is because our ultimate goal is finding a pre-independence relation which generalizes Kaplan and Ramsey's results and the case of simple theories. In other words, $\downarrow^{K^{\downarrow}}$ should coincide with $\downarrow^{K^i}$ over models in NSOP_1 theories, and should also coincide with $\downarrow^f$ over sets in simple theories.

Fact [Kim, 1996]

In simple theory, if $\varphi(x, a)$ divides over a set A, then $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent for all $\downarrow^f$-Morley sequence over A with $a_0 = a$.

Now suppose that $\downarrow^f \Rightarrow \downarrow \Rightarrow \downarrow^{K^f}$ and $\downarrow$ satisfies existence. Then we have $\downarrow^f \Rightarrow \downarrow^{K^{\downarrow}} \Rightarrow \downarrow^{K^i}$. Assume T is NSOP_1 and $M \models T$. First we show that $a \downarrow^{K^i}_M b \rightarrow a \downarrow^{K^{\downarrow}}_M b$. If $\mathrm{tp}(a/Mb)$ $\downarrow$-Kim-forks over M, then it $\downarrow^{K^i}$-Kim-forks over M, and hence it $\downarrow^i$-Kim-forks over M. Now we assume T is simple and A is a set. Suppose $\mathrm{tp}(a/Ab)$ forks over A. Since T is simple, $\downarrow^f$ and $\downarrow^{K^f}$ are equivalent over any set. Thus if some formula $\psi(x, d)$ divides over a set E, then $\psi(x, d_i)\}_{i<\omega}$ is inconsistent for any $\downarrow$-Morley sequence over E with $d_0 = d$. This shows that $\mathrm{tp}(a/Ab)$ $\downarrow$-Kim-forks over A together with the existence of $\downarrow$.

Theorem [Kim, K., Lee, 2023]

Let T be NSOP_1. If $\downarrow$ is stronger than $\downarrow^{K^i}$ and satisfies monotonicity, right extension, strong finite character, and base monotonicity, then $\downarrow^{K^{\downarrow}}$ satisfies existence over sets. In particular, $\downarrow^{K^f}$ satisfies existence over sets.

Question

In NSOP_1 theories, does $\downarrow^{K^{\downarrow^{K^f}}}$ satisfy symmetry or independence theorem over sets?

It is not yet clear whether the conditions in red on the previous page are necessary. The following recently released observations are also worth considering.

Fact [Kruckman, Ramsey, 2023], [Hanson, 2023]

In any theory, $\downarrow^{K^i}$ satisfies existence over sets.

Question

In NSOP_1 theories, does $\downarrow^{K^{\downarrow^{K^i}}}$ satisfy symmetry or independence theorem over sets?

References

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