Defining smallness and independence in arbitrary mathematical structures from a model-theoretic perspective.

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Mathematical structures and languages

In model theory, a language can be likened to a typewriter that consists of

- Built-in keys (quantifiers, logical symbols, variable symblos, and '='):
  $\forall, \exists, \wedge, \vee, \neg, \rightarrow, \leftrightarrow, \ldots, x, y, z, \ldots, x_0, x_1, x_2, \ldots, =$,
- Optionally added keys (non-logical symbols):
  function symbols (e.g. $+, -, \times, ^{-1}, \ldots$) relation symbols (e.g. $<, \leq, \in, \ldots$),
  constant symbols (e.g. $0, 1, \ldots$),

and a formula can be likened to a statement written in a given typewriter (language).

Precisely, a language is defined as a set of some non-logical symbols.
For a given language, say $\mathcal{L}$, we define $\mathcal{L}$-formula as a statement written with $\mathcal{L}$ and built-in symbols.

Example

Let $\mathcal{L} = \{+, -, \times, ^{-1}, 0, 1\}$. Then the following are $\mathcal{L}$-formulas.

- $\forall x_0 x_1 x_2 x_3 (x_0 + x_1 \times y + x_2 \times y \times y + x_3 \times y \times y \times y = 0)$
- $\exists x_0 x_1 x_2 (\neg \exists y (x_0 + x_1 \times y + x_2 \times y \times y = 0)$
- $\forall x (x \neq 0 \rightarrow x \times 1 = x)$

In a given formula, if a variable symbol is not captured by a quantifier, then we say the variable is free. In the first formula, y is a free-variable and x_0 is not a free-variable.

Mathematical structures and languages

Let $\mathcal{L}$ be a language and $\varphi(x)$ an $\mathcal{L}$-formula whose free variable is x. The truth value of $\varphi(x)$ depends on what mathematical structure it is placed in, and what value is assigned to x.

Example

Consider $\mathcal{L} = \{+, -, \times, ^{-1}, 0, 1\}$ and an $\mathcal{L}$-formula $\varphi(x_0, x_1, x_2)$ defined by

$$\neg\exists y(x_0 + x_1 \times y + x_2 \times y \times y = 0).$$

In $\mathbb{R}$, if we assign $1, 0, 1$ to x_0, x_1, x_2, then φ is true. But in $\mathbb{C}$, φ becomes false no matter how we assign values to x_0, x_1, x_2.

Mathematical structures and languages

Let $\mathcal{L}$ be a language, $\mathbb{M}$ be a mathematical structure, and $B \subseteq \mathbb{M}$. By $\mathcal{L}(B)$, we mean an extended language $\mathcal{L} \cup \{b : b \in B\}$, where b is regarded as a constant symbol for each $b \in B$.

Example

Let $\mathcal{L} = \{+, -, \times, ^{-1}, 0, 1\}$ and K an algebraically closed field such that $\mathbb{C} \subsetneq K$. Then the following are $\mathcal{L}(\mathbb{C})$-formulas, but not $\mathcal{L}$-formulas.

- $3 + 21y + 131y^2 = 0$
- $\exists x(\exists y(\sqrt{2} + xy = 0 \wedge 3 + 21y + 131y^2 = 0))$

Indiscernible sequences

In a fixed mathematical structure $\mathbb{M}$ and a language $\mathcal{L}$, an indiscernible sequence is a sequence $(a_i)_{i<\mathbb{N}}$ in $\mathbb{M}$ that has some sort of "consistent tendency" with respect to $\mathcal{L}$. Precisely, we say a sequence $(a_i)_{i<\mathbb{N}}$ is indiscernible over a set $A \subseteq \mathbb{M}$ with respect to $\mathcal{L}$ in $\mathbb{M}$ if

$$a_{i_0}...a_{i_{n-1}} \equiv_A^{\mathcal{L}} a_{j_0}...a_{j_{n-1}}$$

for all $i_0 < \cdots < i_{n-1}$ and $j_0 < \cdots < j_{n-1}$ in $\mathbb{N}$.

It means that those two finite sequences satisfy exactly the same formulas in $\mathcal{L}(A)$.

In other words, it means that if $\varphi(a_{i_0}, \cdots, a_{i_{n-1}})$ is true (in $\mathbb{M}$) for some $\mathcal{L}(A)$-formula, then $\varphi(a_{j_0}, \cdots, a_{j_{n-1}})$ is also true.

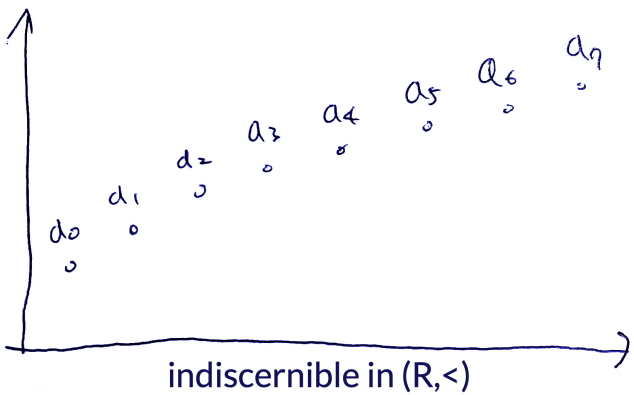
In other words, it means that if some phenomenon occurs in the tuple on the left hand side, and that phenomenon can be written with $\mathcal{L}(A)$, then the same phenomenon occurs in the tuple on the right hand side.

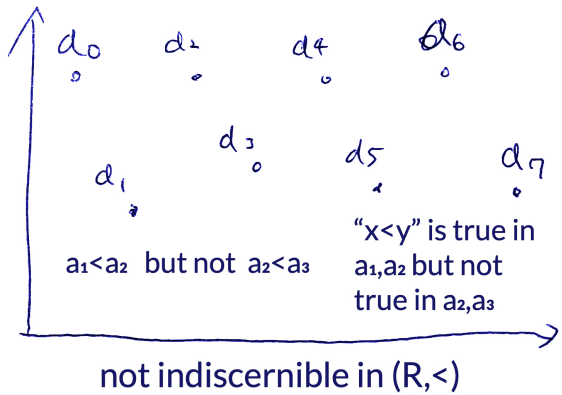
Indiscernible sequences

Example

Consider $\mathcal{L} = \{<\}$ and $\mathbb{R}$.

- Monotonically increasing/decreasing sequences are indiscernible over $\emptyset$.
- If a sequence is oscillating, then in is not indiscernible.





The second sequence is not indiscernible. Consider $\varphi(x, y, z) := x < y < z$ and $i_0 = 1, i_1 = 3, i_2 = 4, j_0 = 0, j_1 = 3, j_2 = 5$. Then $\varphi(a_{i_0}, a_{i_1}, a_{i_2})$ is true but $\varphi(a_{j_0}, a_{j_1}, a_{j_2})$ is false.

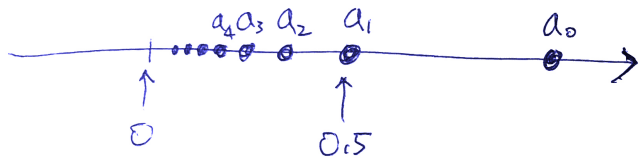
Indiscernible sequences

The base set of a given indiscernible sequence (A in the above definition) can be regarded as an "observer". The same sequence may or may not be an indiscernible sequence, depending on how we choose the base set.

Example

In $(\mathbb{R}, <)$, let $(a_n) = 1/n$ for each $n < \omega$. Then $(a_n)_{n<\omega}$ is indiscernible over $\{0\}$ but not indiscernible over $\{0.5\}$.

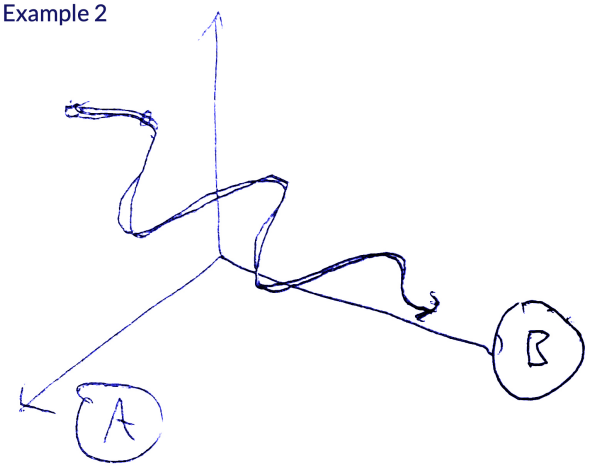
Example 1

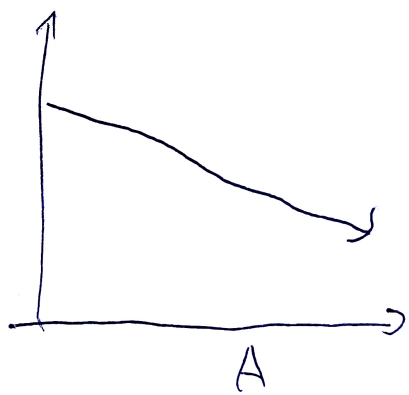


$a_0 \not\equiv_{\{0.5\}} a_2$

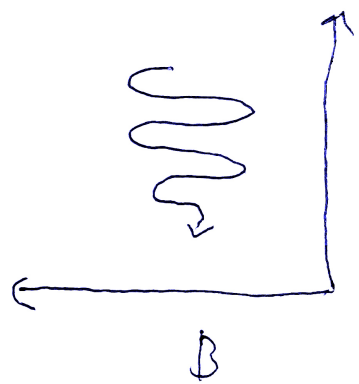
$a_0 \models 0.5 < x$ but $a_2 \not\models 0.5 < x$

Example 2





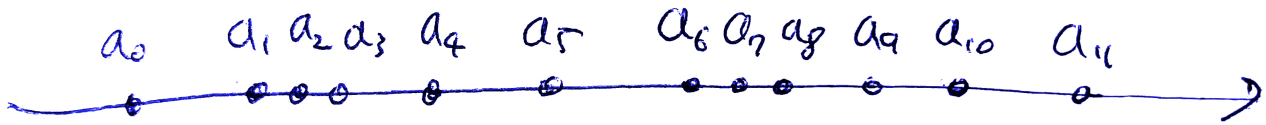
indiscernible over A



not indiscernible over B

Indiscernible sequences

If the language $\mathcal{L}$ becomes richer and can express a wider variety of movements of a sequence, then the sequences become harder to be indiscernible.



indiscernible in (R,<) but not indiscernible in (R,<,d(x,y))
where d(x,y) is the distance between x and y

Let $\mathcal{L} = \{<\}$ and $\mathcal{L}' = \{<, d\}$, where d is a distance function. Then the sequence above is not indiscernible since $a_2 a_3 a_4 \not\equiv a_5 a_6 a_7$. Consider $\varphi(x, y, z) := x < y < z \wedge (d(x, y) < d(y, z))$.

Definable sets

Let $\mathcal{L}$ be a language and $\mathbb{M}$ a mathematical structure. We say a subset $D \subseteq \mathbb{M}^n$ is definable if there are $(a_0, ..., a_{m-1}) \in \mathbb{M}^m$ and an $\mathcal{L}$-formula $\varphi(x_0, ..., x_{n-1}, y_0, ..., y_{m-1})$ such that

$(d_0, ..., d_{n-1}) \in D$ if and only if $\varphi(d_0, ..., d_{n-1}, a_0, ..., a_{m-1})$ is true in $\mathbb{M}$.

In the above definition, we call $(a_0, ..., a_{m-1})$ a parameter.

Example

Let $\mathcal{L} = \{+, -, \times, ^{-1}, 0, 1\}$. Then the set of all $(x, y) \in \mathbb{R}^2$ whose distance from (a, b) is less than or equal to c is definable for any real numbers a, b, c. ($\varphi(x, y, z, u, v) := \exists w(v \times v - (x - z) \times (x - z) - (y - u) \times (y - u) = w \times w)$ defines the set with $z = a, u = b, v = c$.)

Smallness

Idea

- Consider a family of definable sets $\{D_i\}_{i\in\mathbb{N}}$ that is uniformly defined by an $\mathcal{L}$-formula $\varphi(\bar{x},\bar{y})$ and a family of parameters $\{\bar{b}_i\}_{i\in\mathbb{N}}$ (i.e. for each $i\in\mathbb{N}$, $\bar{d}\in D_i$ if and only if $\varphi(\bar{d},\bar{b}_i)$ is true), and the possibility that the definable sets have an intersection (i.e. the possibility that $\{\varphi(\bar{x},\bar{b}_i)\}_{i\in\mathbb{N}}$ has a common solution).
- If the parameters $\{\bar{b}_i\}_{i\in\mathbb{N}}$ were chosen randomly and in a jumbled way, it would be less likely that $\{\varphi(\bar{x},\bar{b}_i)\}_{i\in\mathbb{N}}$ would have a common solution.
- On the other hand, if $\{\bar{b}_i\}_{i\in\mathbb{N}}$ were chosen to have some tendency or order, the possibility of $\{\varphi(\bar{x},\bar{b}_i)\}_{i\in\mathbb{N}}$ having a common solution would increase.
- So if we choose $\{\bar{b}_i\}_{i\in\mathbb{N}}$ as an indiscernible sequence, then we can expect $\{\varphi(\bar{x},\bar{b}_i)\}_{i\in\mathbb{N}}$ to be more likely to have a common solution.
- If $\{\varphi(\bar{x},\bar{b}_i)\}_{i\in\mathbb{N}}$ has no intersection even though $(b_i)_{i\in\mathbb{N}}$ is indiscernible, then we may consider $\varphi(\bar{x},\bar{b})$ to be 'small', or to satisfy some property that can be metaphorically called 'smallness'.

Smallness

Definition

We say a formula $\varphi(\bar{x}, \bar{a})$ divides over a set A if there exists an indiscernible sequence $(\bar{a}_i)_{i<\omega}$ over A with $\bar{a}_0 = \bar{a}$ such that $\{\varphi(\bar{x}, \bar{a}_i) : i \in \mathbb{N}\}$ has no common solution.

$$\varphi(x, a) \text{ divides} \approx \text{the set defined by } \varphi(x, a) \text{ is 'small'}$$

Example

Let $\mathcal{L} = \{+, -, \times, ^{-1}, 0, 1\}$ and consider a family of definable sets $\{D_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}^2$ defined by $\{\varphi(x, y, a_i, b_i, c_i)\}_{i\in\mathbb{N}}$ where the sequence $(a_i b_i c_i)_{i\in\mathbb{N}}$ is indiscernible over $\emptyset$, and

$$\varphi(x, y, z, u, v) := \exists w(v \times v - (x - z) \times (x - z) - (y - u) \times (y - u) = w \times w).$$

It can be checked that if a sequence in $\mathbb{R}$ (or a saturated elementary extension of $\mathbb{R}$) is indiscernible, then all members in the sequence are living in the same Dedekind cut. So $\{D_i\}_{i\in N}$ is a set of unit balls, whose centers and radius are the same. Thus $\varphi(x, y, a_0, b_0, c_0)$ can not divide over $\emptyset$.

Independence

This 'smallness' gives a notion of independence as follows.

Definition [Non-dividing independence]

Fix a language $\mathcal{L}$ and a mathematical structure $\mathbb{M}$. We write $a \downarrow^d_C b$ if there is no dividing formula $\varphi(x) \in \mathcal{L}(Cb)$ over C such that $\varphi(a)$ is true in $\mathbb{M}$. If $a \downarrow^d_C b$, we say a is non-dividing independent from b over C.

Idea

$a \downarrow^d_C b$

$\Rightarrow$ There is no dividing ('small') formula $\varphi(x) \in \mathcal{L}(Cb)$ over C capturing a.

$\Rightarrow$ a is 'relatively free' from b over C.

$\Rightarrow$ We may consider a to be 'independent' from b over C.

Independence

Fact

Let $K \subseteq L$ be algebraically closed fields and $\mathcal{L} = \{+, -, \times, ^{-1}, 0, 1\}$. Suppose there is a sequence $(a_i)_{i \in \mathbb{N}}$ in L such that $a_i \notin \overline{K(a_{<i})}$ for each $i \in \mathbb{N}$, where $a_{<i} = \{a_j : j < i\}$. Then $(a_i)_{i \in \mathbb{N}}$ is an indiscernible sequence over K.

Example

Let $\mathcal{L} = \{+, -, \times, ^{-1}, 0, 1\}$ and $K \subseteq L$ be algebraically closed fields and suppose L is sufficiently saturated (large). If $a \downarrow^d_K b$ for some $a, b \in L \setminus K$, then a and b are algebraically independent over K.

proof Suppose that a, b are algebraically dependence over K. Then there is a polynomial $f(x, y)$ over K such that $f(a, b) = 0$. If we define an $\mathcal{L}(K)$-formula $\varphi(x, y)$ by $f(x, y) = 0$, then $\varphi(x, b)$ is an $\mathcal{L}(Kb)$-formula and $\varphi(a, b)$ is true. Since we assume L is saturated (large), we can find a sequence $(b_i)_{i \in \mathbb{N}}$ with $b_0 = b$ such that $b_i \notin \overline{K(b_{<i})}$ for each $i \in \mathbb{N}$. This sequence is indiscernible over K. It is enough to show that $\{\varphi(x, b_i)\}_{i \in \mathbb{N}}$ has no common solution. If $\{\varphi(x, b_i)\}_{i \in \mathbb{N}}$ has a common solution, say a', then $f(a', b_i) = 0$ for each $i \in \mathbb{N}$. Since $f(a', y) = 0$ has only finitely many solutions, there are $i \neq j \in \mathbb{N}$ such that $b_i = b_j$, which yield a contradiction.

Fact [non-dividing independence in fields]

In an algebraically closed field, $a \downarrow^d_C b$ if and only if a and b are algebraically independent over C.

Independence

Let $\mathbb{G}$ be a graph and $\mathcal{L} = \{E\}$ a language where E is the edge relation. For $A, B \subseteq \mathbb{G}$, we write $E(A, B)$ if $E(a, b)$ for all $a \in A$, $b \in B$. For $A, B \subseteq \mathbb{G}$ with $A \cap B = \emptyset$, we write $K_n(A/B)$ if there exists $A_0 \subseteq AB$ with $|A_0| = n$ such that $A_0 \cap A \neq \emptyset \neq A_0 \cap B$ and $A_0 \cap C$ and $R(A_0 \cap A, a_0 \cap B)$.
Let $\mathcal{H}_n$ be the countable generic K_n-free graph. That is, $\mathcal{H}_n$ is the unique countable graph such that

- every finite K_n-free graph can be embedded into $\mathcal{H}_n$
- every isomorphism between two finite subgraphs $A, B \subseteq \mathcal{H}_n$ can be extended to an automorphism of $\mathcal{H}_n$.

Let $\mathbb{H}_n$ be a saturated elementary extension of $\mathcal{H}_n$ with respect to $\mathcal{L} = \{E\}$.

Fact [non-dividing independence in graphs]

In $\mathbb{H}_n$, $A \downarrow^d_C B$ if and only if $A \cap B \subseteq C$ and for all $b \subseteq B \setminus C$, $K_n(b/AC)$ implies $K_n(b/C)$.

Ideals, smallness, independence

A family of definable sets $\mathcal{I}$ is called an ideal if

- $D, E \in \mathcal{I} \Rightarrow D \cup E \in \mathcal{I}$,
- $D \in \mathcal{I}$ and $E \subseteq D \Rightarrow E \in \mathcal{I}$.

Let $\mathcal{L}$ be a language $\mathbb{M}$ a structure. For a given family of ideals $\mathcal{I} = \{\mathcal{I}_A\}_{A \subseteq \mathbb{M}}$, we define notions of smallness and independence as follows.

Definition

A definable set D is $\mathcal{I}$-small over A if $D \in \mathcal{I}_A$.

Definition

We say a is $\mathcal{I}$-independent from b over C and write $a \mathop{\downarrow}^{\mathcal{I}}_{C} b$ if there is no $\mathcal{L}(Cb)$ formula $\varphi(x)$ such that $\varphi(x)$ is $\mathcal{I}$-small over C and $\varphi(a)$ is true.

Ideals, smallness, independence

Conversely, For a given invariant ternary relation, say $\downarrow$, we can define a family of ideals as follows.

Definition

- We say a sequence $(a_i)_{i\in\mathbb{N}}$ is a $\downarrow$-Morley sequence over a set B if it is indiscernible over B and $a_i \downarrow_B a_{<i}$ for all $i \in \mathbb{N}$.
- We say a formula $\varphi(x, a)$ $\downarrow$-Kim-divides over B if there is a $\downarrow$-Morley sequence $(a_i)_{i\in\mathbb{N}}$ with $a_0 = a$ over B such that $\{\varphi(x, a_i)\}_{i\in\mathbb{N}}$ has no common solution.
- We say a formula $\varphi(x, a)$ $\downarrow$-Kim-forks over B if the definable set D defined by $\varphi(x, a)$ is a subset of $E_0, ..., E_{n-1}$ where E_i is defined by some $\downarrow$-Kim-dividing formula over B for each $i < n$.
- $\mathcal{I}_A^{\downarrow} = \{D \subseteq \mathbb{M} : D$ is defined by some $\downarrow$-Kim-forking formula over $A\}$ forms an ideal for each $A \subseteq \mathbb{M}$.

A given independence relation $\downarrow$ induces a family of ideals $\mathcal{I} = \{\mathcal{I}_A^{\downarrow}\}_{A\subseteq\mathbb{M}}$, and that family of ideals gives a new notion of independence relation $\downarrow^{(\mathcal{I}^{\downarrow})}$.

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