Existence of pre-independence relations induced by Morley sequences

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Indiscernible sequences

In a fixed mathematical structure $\mathbb{M}$ and a language $\mathcal{L}$, an indiscernible sequence is a sequence $(a_i)_{i<\mathbb{N}}$ in $\mathbb{M}$ that has some sort of "consistent tendency" with respect to $\mathcal{L}$. Precisely, we say a sequence $(a_i)_{i<\mathbb{N}}$ is indiscernible over a set $A \subseteq \mathbb{M}$ with respect to $\mathcal{L}$ in $\mathbb{M}$ if

$$a_{i_0} \ldots a_{i_{n-1}} \equiv_A^{\mathcal{L}} a_{j_0} \ldots a_{j_{n-1}}$$

for all $i_0 < \cdots < i_{n-1}$ and $j_0 < \cdots < j_{n-1}$ in $\mathbb{N}$.

It means that those two finite sequences satisfy exactly the same formulas in $\mathcal{L}(A)$.

In other words, it means that if $\varphi(a_{i_0}, \cdots, a_{i_{n-1}})$ is true (in $\mathbb{M}$) for some $\mathcal{L}(A)$-formula, then $\varphi(a_{j_0}, \cdots, a_{j_{n-1}})$ is also true.

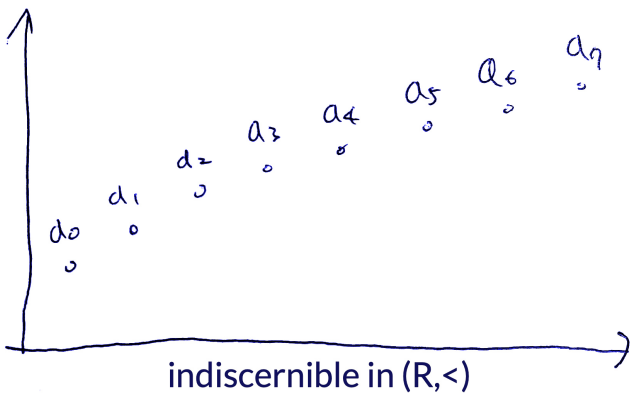
Some phenomenon occurs in the tuple on the left hand side, and that phenomenon can be written with $\mathcal{L}(A)$, then the same phenomenon occurs in the tuple on the right hand side.

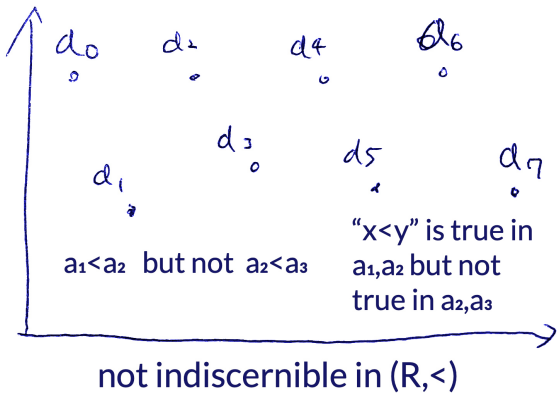
Indiscernible sequences

Example

Consider $\mathcal{L} = \{<\}$ and $\mathbb{R}$.

- Monotonically increasing/decreasing sequences are indiscernible over $\emptyset$.
- If a sequence is oscillating, then in is not indiscernible.





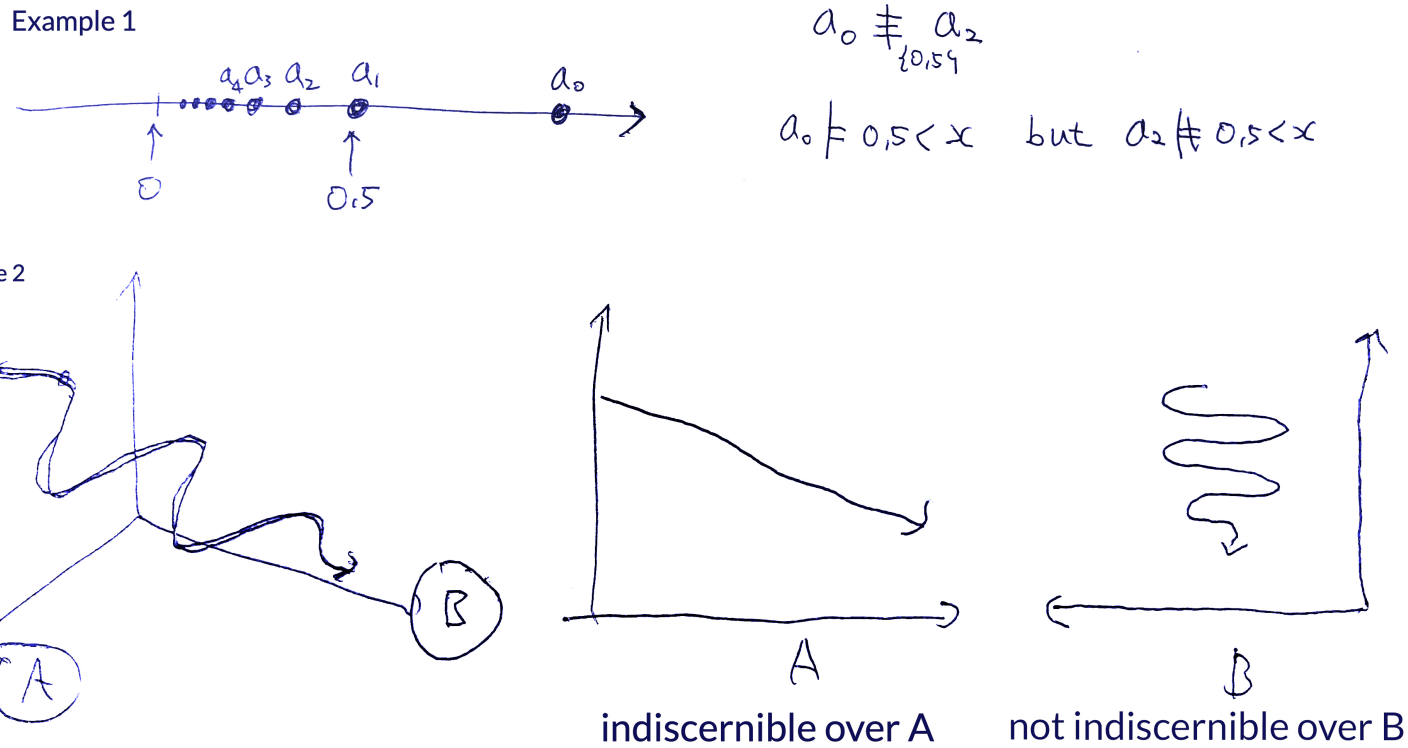
The second sequence is not indiscernible. Consider $\varphi(x, y, z) := x < y < z$ and $i_0 = 1, i_1 = 3, i_2 = 4, j_0 = 0, j_1 = 3, j_2 = 5$. Then $\varphi(a_{i_0}, a_{i_1}, a_{i_2})$ is true but $\varphi(a_{j_0}, a_{j_1}, a_{j_2})$ is false.

Indiscernible sequences

The base set of a given indiscernible sequence (A in the above definition) can be regarded as an "observer". The same sequence may or may not be an indiscernible sequence, depending on how we choose the base set.

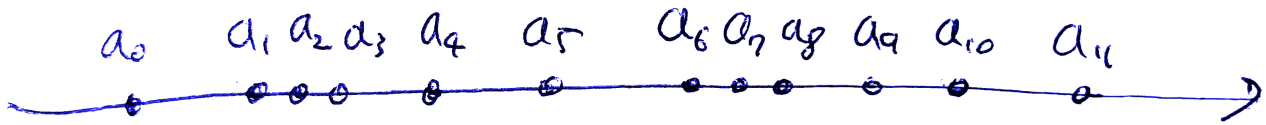
Example

In $(\mathbb{R}, <)$, let $(a_n) = 1/n$ for each $n < \omega$. Then $(a_n)_{n<\omega}$ is indiscernible over $\{0\}$ but not indiscernible over $\{0.5\}$.



Indiscernible sequences

If the language $\mathcal{L}$ becomes richer and can express a wider variety of movements of a sequence, then the sequences become harder to be indiscernible.



indiscernible in (R,<) but not indiscernible in (R,<,d(x,y))
where d(x,y) is the distance between x and y

Let $\mathcal{L} = \{<\}$ and $\mathcal{L}' = \{<, d\}$, where d is a distance function. Then the sequence above is not indiscernible since $a_2 a_3 a_4 \not\equiv a_5 a_6 a_7$. Consider $\varphi(x, y, z) := x < y < z \wedge (d(x, y) < d(y, z))$.

Definable sets

Let $\mathcal{L}$ be a language and $\mathbb{M}$ a mathematical structure. We say a subset $D \subseteq \mathbb{M}^n$ is definable if there are $(a_0, ..., a_{m-1}) \in \mathbb{M}^m$ and an $\mathcal{L}$-formula $\varphi(x_0, ..., x_{n-1}, y_0, ..., y_{m-1})$ such that

$(d_0, ..., d_{n-1}) \in D$ if and only if $\varphi(d_0, ..., d_{n-1}, a_0, ..., a_{m-1})$ is true in $\mathbb{M}$.

In the above definition, we call $(a_0, ..., a_{m-1})$ a parameter.

Example

Let $\mathcal{L} = \{+, -, \times, ^{-1}, 0, 1\}$. Then the set of all $(x, y) \in \mathbb{R}^2$ whose distance from (a, b) is less than or equal to c is definable for any real numbers a, b, c. ($\varphi(x, y, z, u, v) := \exists w(v \times v - (x - z) \times (x - z) - (y - u) \times (y - u) = w \times w)$ defines the set with $z = a, u = b, v = c$.)

Smallness

Idea

- Consider a family of definable sets $\{D_i\}_{i\in\mathbb{N}}$ that is uniformly defined by an $\mathcal{L}$-formula $\varphi(\bar{x},\bar{y})$ and a family of parameters $\{\bar{b}_i\}_{i\in\mathbb{N}}$ (i.e. for each $i\in\mathbb{N}$, $\bar{d}\in D_i$ if and only if $\varphi(\bar{d},\bar{b}_i)$ is true), and the possibility that the definable sets have an intersection (i.e. the possibility that $\{\varphi(\bar{x},\bar{b}_i)\}_{i\in\mathbb{N}}$ has a common solution).
- If the parameters $\{\bar{b}_i\}_{i\in\mathbb{N}}$ were chosen randomly and in a jumbled way, it would be less likely that $\{\varphi(\bar{x},\bar{b}_i)\}_{i\in\mathbb{N}}$ would have a common solution.
- On the other hand, if $\{\bar{b}_i\}_{i\in\mathbb{N}}$ were chosen to have some tendency or order, the possibility of $\{\varphi(\bar{x},\bar{b}_i)\}_{i\in\mathbb{N}}$ having a common solution would increase.
- So if we choose $\{\bar{b}_i\}_{i\in\mathbb{N}}$ as an indiscernible sequence, then we can expect $\{\varphi(\bar{x},\bar{b}_i)\}_{i\in\mathbb{N}}$ to be more likely to have a common solution.
- If $\{\varphi(\bar{x},\bar{b}_i)\}_{i\in\mathbb{N}}$ has no intersection even though $(b_i)_{i\in\mathbb{N}}$ is indiscernible, then we may consider $\varphi(\bar{x},\bar{b})$ to be 'small', or to satisfy some property that can be metaphorically called 'smallness'.

Smallness

Definition

We say a formula $\varphi(\bar{x}, \bar{a})$ divides over a set A if there exists an indiscernible sequence $(\bar{a}_i)_{i<\omega}$ over A with $\bar{a}_0 = \bar{a}$ such that $\{\varphi(\bar{x}, \bar{a}_i) : i \in \mathbb{N}\}$ has no common solution.

$$\varphi(x, a) \text{ divides} \approx \text{the set defined by } \varphi(x, a) \text{ is 'small'}$$

Example

Let $\mathcal{L} = \{+, -, \times, ^{-1}, 0, 1\}$ and $\mathbb{M}$ be a saturated model of $\mathrm{Th}_{\mathcal{L}}(\mathbb{R})$. consider a family of definable sets $\{D_i\}_{i\in\mathbb{N}}$ in $\mathbb{M}^2$ defined by $\{\varphi(x, y, a_i, b_i, c_i)\}_{i\in\mathbb{N}}$ where the sequence $(a_i b_i c_i)_{i\in\mathbb{N}}$ is indiscernible over $\emptyset$, and

$$\varphi(x, y, z, u, v) := \exists w(v \times v - (x - z) \times (x - z) - (y - u) \times (y - u) = w \times w).$$

It can be checked that if a sequence in $\mathbb{M}$ is indiscernible, then all members in the sequence are living in the same Dedekind cut. So $\{D_i\}_{i\in N}$ is a set of unit balls, whose centers and radius are the same. Thus $\varphi(x, y, a_0, b_0, c_0)$ can not divide over $\emptyset$ if there is a rational number r such that $|r| < |c_0|$.

Forking

In order to implement the properties we would expect from a concept named 'smallness', we can introduce a notion of 'forking', which is a slight extension of the concept of dividing.

Definition

We say a formula $\varphi(x)$ forks over a set A if there are dividing formulas $\psi_0(x, b_0), ..., \psi_n(x, b_n)$ over A such that $\varphi(x) \vdash \bigvee_{i \leq n} \psi_i(x, b_i)$.
Note that every dividing formula is a forking formula, hence every definable set defined by a dividing formula is defined by a forking formula.

The family of definable sets that defined by forking formulas satisfies the following two property:

- If D is a definable set defined by a forking formula and $E \subseteq D$ is a definable set, then E is also definable by a forking formula.
  (If D is 'small' and $E \subseteq D$, then E is also small.)
- If D, E are definable sets defined by forking formulas, then $D \cup E$ is also definable set defined by forking formula.
  (Every union of finitely many small sets is also small.)

Independence

This 'smallness' gives a notion of independence as follows.

Definition [Non-forking independence]

Fix a language $\mathcal{L}$ and a mathematical structure $\mathbb{M}$. We write $a \downarrow^f_C b$ if there is no forking formula $\varphi(x) \in \mathcal{L}(Cb)$ over C such that $\varphi(a)$ is true in $\mathbb{M}$. If $a \downarrow^f_C b$, we say a is non-forking independent from b over C.

Idea

$a \downarrow^f_C b$

$\Rightarrow$ There is no forking ('small') formula $\varphi(x) \in \mathcal{L}(Cb)$ over C capturing a.

$\Rightarrow$ a is 'relatively free' from b over C.

$\Rightarrow$ We may consider a to be 'independent' from b over C.

Example

In an algebraically closed field, $a \downarrow^f_C b$ if and only if a and b are algebraically independent over C.

Pre-independence relations

Definition

A *pre-independence relation* $\downarrow$ is an invariant ternary relation on sets. If $(A, B, C) \in \downarrow$, then we write $A \downarrow_C B$ and say *A is $\downarrow$-independent from B over C*. The following are the properties we consider for a pre-independence relation:

- Monotonicity over C: If $AA' \downarrow_C BB'$, then $A \downarrow_C B$.
- Base monotonicity over C: If $A \downarrow_C BB'$, then $A \downarrow_{CB} B'$.
- Transitivity over D: If $A \downarrow_{DB} C$ and $B \downarrow_D C$, then $AB \downarrow_D C$.
- Right extension over C: If $A \downarrow_C B$, then for all D, there exists $D' \equiv_{CB} D$ such that $A \downarrow_C BD'$.
- Existence over C: $A \downarrow_C \emptyset$ for all A.
- Symmetry over C: If $A \downarrow_C B$, then $B \downarrow_C A$.
- Finite character over C: If $A \not\downarrow_C B$, then there exist finite $a \subseteq A$ and $b \subseteq B$ such that $a \not\downarrow_C b$.

Note that non-forking independence $\downarrow^f$ is a special case of pre-independence relation.

Morley sequences and $\downarrow$-Kim-independence

One can generalize the concept of forking using pre-independence relations.

Definition

- We say a sequence $(a_i)_{i\in\mathbb{N}}$ is a $\downarrow$-Morley sequence over a set B if it is indiscernible over B and $a_i \downarrow_B a_{<i}$ for all $i \in \mathbb{N}$.
- We say a formula $\varphi(x, a)$ $\downarrow$-Kim-divides over B if there is a $\downarrow$-Morley sequence $(a_i)_{i\in\mathbb{N}}$ with $a_0 = a$ over B such that $\{\varphi(x, a_i)\}_{i\in\mathbb{N}}$ has no common solution.
- We say a formula $\varphi(x, a)$ $\downarrow$-Kim-forks over B if the definable set D defined by $\varphi(x, a)$ is a subset of $E_0, ..., E_{n-1}$ where E_i is defined by some $\downarrow$-Kim-dividing formula over B for each $i < n$.
- We write $a \downarrow^{K^{\downarrow}}_C b$ and say a is non-$\downarrow$-Kim-forking independent from b over C if there is no $\downarrow$-Kim-forking formula $\varphi(x) \in \mathcal{L}(Cb)$ over C such that $\varphi(a)$ is true in $\mathbb{M}$.

Note that being a Morley sequence is more difficult than being an indiscernible sequence. So non-forking independence is stronger than non-$\downarrow$-Kim-independence (i.e., $a \downarrow^f_C b$ implies $a \downarrow^{K^{\downarrow}}_C b$).

Existence of $\downarrow$-Kim-independence

In model theory, theories can be classified according to the possibility of existence of a family of definable sets with certain combinatoric patterns.

Definition

Let L be a language, T an L-theory, $\mathbb{M}$ a monster model of T.
T has *TP* if there exist an L-formula $\varphi(x,y)$, $k<\omega$, and a tree-indexed set of tuples of parameters $(a_\eta)_{\eta\in\omega^{<\omega}}$ in $\mathbb{M}$ such that

- $\{\varphi(x,a_{\eta_0}),...,\varphi(x,a_{\eta_n})\}$ is consistent if $\eta_0 \trianglelefteq \cdots \trianglelefteq \eta_n$,
- $\{\varphi(x,a_{\eta^\frown\langle i_0\rangle}),...,\varphi(x,a_{\eta^\frown\langle i_k\rangle})\}$ is inconsistent for all $\eta\in\omega^{<\omega}$ and $i_0<\cdots<i_k<\omega$.

T has *SOP*$_1$ if there exist an L-formula $\varphi(x,y)$, $k<\omega$, and a tree-indexed set of tuples of parameters $(a_\eta)_{\eta\in\omega^{<\omega}}$ in $\mathbb{M}$ such that

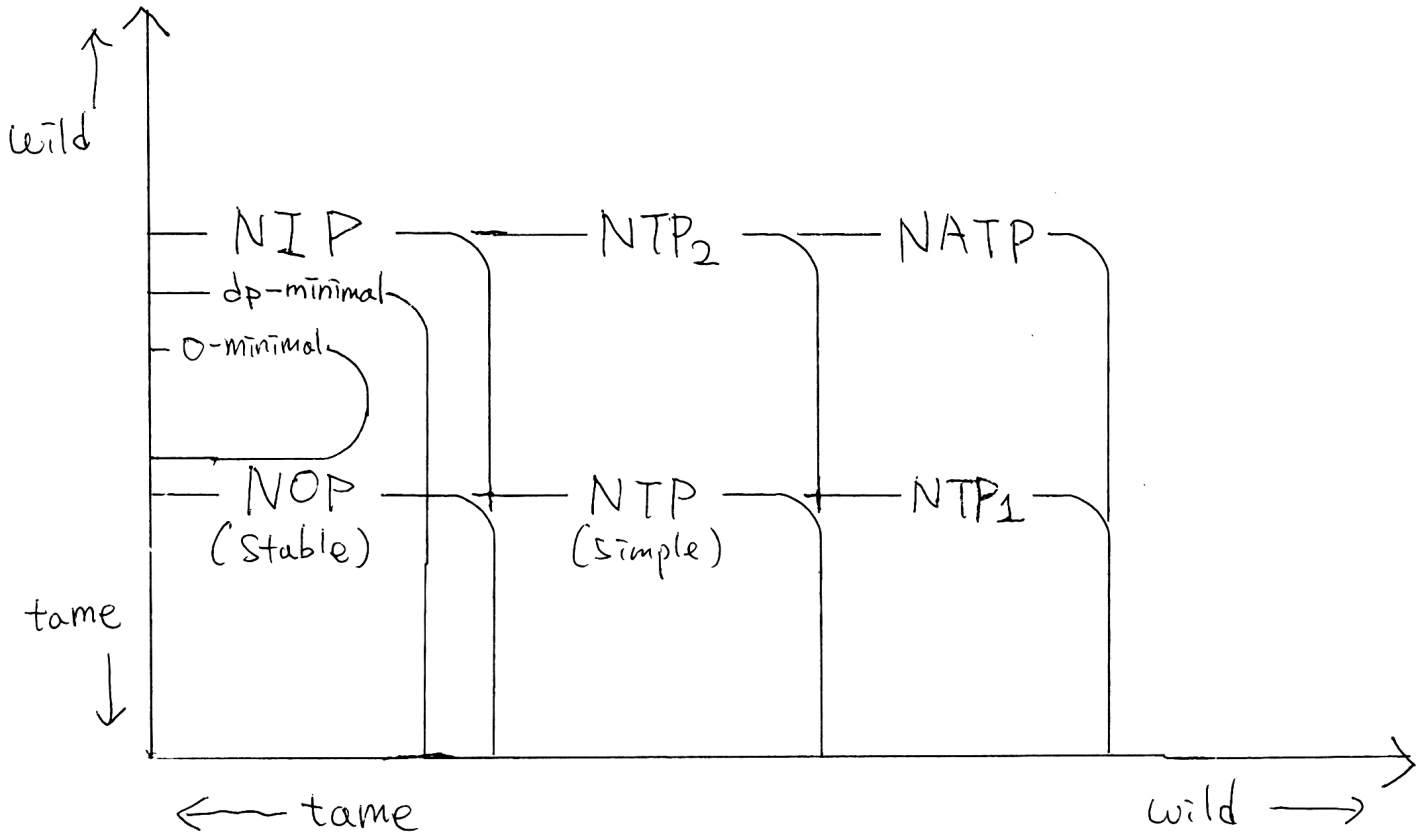
- $\{\varphi(x,a_{\eta_0}),...,\varphi(x,a_{\eta_n})\}$ is consistent if $\eta_0 \trianglelefteq \cdots \trianglelefteq \eta_n$,
- $\{\varphi(x,a_{\eta^\frown\langle 1\rangle}),\varphi(x,a_\nu)\}$ is inconsistent if $\eta^\frown\langle 0\rangle \trianglelefteq \nu$.

T is *NTP* (*NSOP*$_1$) if it does not have TP (SOP$_1$, respectively). We say T is *simple* if it is NTP.

Fact

If T is NTP, then it is NSOP$_1$. The converse does not hold in general.

Classification theory



Kim-Pillay style criterion

Pre-independence relation can give a new characterization of some dividing line.

Fact [Kim, Pillay, 1997]

A theory T is NTP if and only if there exists a pre-independence relation $\downarrow$ satisfying monotonicity, existence, right extension, symmetry, base monotonicity, transitivity, finite character, and local character over sets, and the independence theorem over models.
Moreover, if such $\downarrow$ exists, then $\downarrow = \downarrow^f$.

Fact [Kaplan, Ramsey, 2017]

A theory T is NSOP_1 if and only if there exists a pre-independence relation $\downarrow$ satisfying monotonicity, existence, right extension, symmetry, strong finite character, and independence theorem over models.

Moreover, if such $\downarrow$ exists, then $\downarrow \Rightarrow \downarrow^{K^{\downarrow^f}}$ over models.

Question

Can we find an independent notion that works well in NSOP_1, not only over models, but also over an arbitrary set?

It is known that in NSOP_1 theories, if $\mathop{\smile\!\!\!\!\!\!|}^{f}$ satisfies existence, then $\mathop{\smile\!\!\!\!\!\!|}^{K^{\mathop{\smile\!\!\!\!\!\!|}^{f}}}$ satisfies existence, right extension, symmetry, strong finite character, independence theorem over an arbitrary set. But it is not yet known whether $\mathop{\smile\!\!\!\!\!\!|}^{f}$ satisfies existence in NSOP_1 generally.

$\mathop{\smile\!\!\!\!\!\!|}^{f}$ is actually a pretty strong pre-independence relation and the stronger the pre-independence relation, the less likely it is to satisfy existence.

Theorem [Kim, K., Lee, 2024]

In NSOP_1, over an arbitrary set, $\mathop{\smile\!\!\!\!\!\!|}^{K^{\mathop{\smile\!\!\!\!\!\!|}^{f}}}$ satisfies existence.

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