Existence of Kim-independence in NSOP_1 theories over sets

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We fix a complete theory T in a language L,
a sufficiently saturated model $\bar{M}$ of T.

Def. Let $\varphi(x,b) \in L(\bar{M})$ and $C \subseteq \bar{M}$.

(i) We say $\varphi(x,b)$ divides over C if there is an indiscernible sequence $(b_i)_{i\in\omega}$ over C with $b_0 = b$ such that $\{\varphi(x,b_i)\}_{i\in\omega}$ is inconsistent.

(ii) We say $\varphi(x,b)$ forks over C if there are dividing formulas $\psi_0(x,c_0), \cdots, \psi_n(x,c_n)$ over C such that $\varphi(x,b) \vdash \bigvee \psi_i(x,b_i)$.

(iii) We say a complete type p(x) divides (forks) over C if it contains a dividing formula (forking formula).

(iv) We write $a \overset{d}{\underset{C}{\perp}} b$ if $tp(a/Cb)$ does not divide over C.

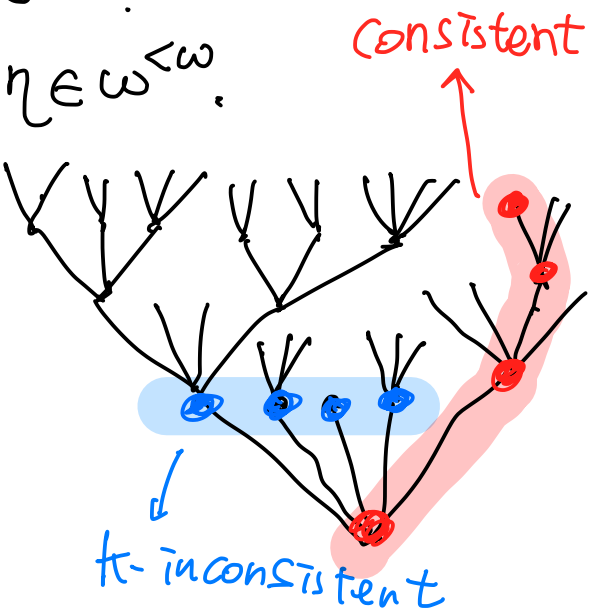
(v) We write $a \overset{f}{\underset{C}{\perp}} b$ if $tp(a/Cb)$ does not fork over C.

Def. We say T is simple if there is no $\varphi(x,y)\in L$ and a tree indexed set of parameters $(b_\eta)_{\eta\in\omega^{<\omega}}$ and a natural number k such that

(i) $\{\varphi(x,b_\eta):\eta\lhd\nu\}$ is consistent for all $\nu\in\omega^\omega$.

(ii) $\{\varphi(x,b_{\eta^\frown\langle i\rangle}):i\in\omega\}$ is k-inconsistent for all $\eta\in\omega^{<\omega}$.

If there are a formula and an indexed set of parameters satisfying (i) and (ii), then we call the pair of them a witness of tree property.



Fact. [Kim, Pillay] In simple theories, $\mathcal{L}^d\simeq\mathcal{L}^f$.

Def. By a pre-independence relation, we mean an invariant ternary relation of small sets. Let $\downarrow$ be a pre-independence relation. We say $\downarrow$ satisfies

(i) monotonicity if $aa' \underset{C}{\downarrow} bb'$ implies $a \underset{C}{\downarrow} b$.

(ii) existence if $a \underset{C}{\downarrow} \emptyset$ for all a and C,

(iii) right extension if $\left[a \underset{C}{\downarrow} b \wedge d \subseteq \mathcal{M} \Rightarrow \exists d' \equiv_{Cb} d \text{ s.t. } a \underset{C}{\downarrow} bd' \right]$,

(iv) symmetry if $a \underset{C}{\downarrow} b$ implies $b \underset{C}{\downarrow} a$,

(v) base monotonicity if $a \underset{C}{\downarrow} bd$ implies $a \underset{Cd}{\downarrow} b$,

(vi) transitivity if $a \underset{Cb}{\downarrow} d \wedge b \underset{C}{\downarrow} d$ implies $ab \underset{C}{\downarrow} d$.

We also consider the following properties for pre-independence relations.

(vii) Strong finite character: if $a \not\downarrow_C b$, then there is $\varphi(x,b) \in tp(a/Cb)$ such that $\alpha \not\downarrow_C b$ for all $\alpha \models \varphi(x,b)$.

(viii) Local character: for all a and C, there is $B \subseteq C$ with $|B| \leq |T|$ such that $a \downarrow_B C$.

(ix) Independence theorem over models: Let $M \prec \widetilde{M}$. If $a \downarrow_M b$, $a' \downarrow_M c$, $b \downarrow_M c$, and $a \equiv_M a'$, then $\exists a''$ such that $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a'$, and $a'' \downarrow_M bc$.

Simplicity for complete theories can be characterized by the existence of certain pre-independence relations.

Fact. [Kim, Pillay] T is simple if and only if there is a pre-independence relation satisfying (i)—(ix).

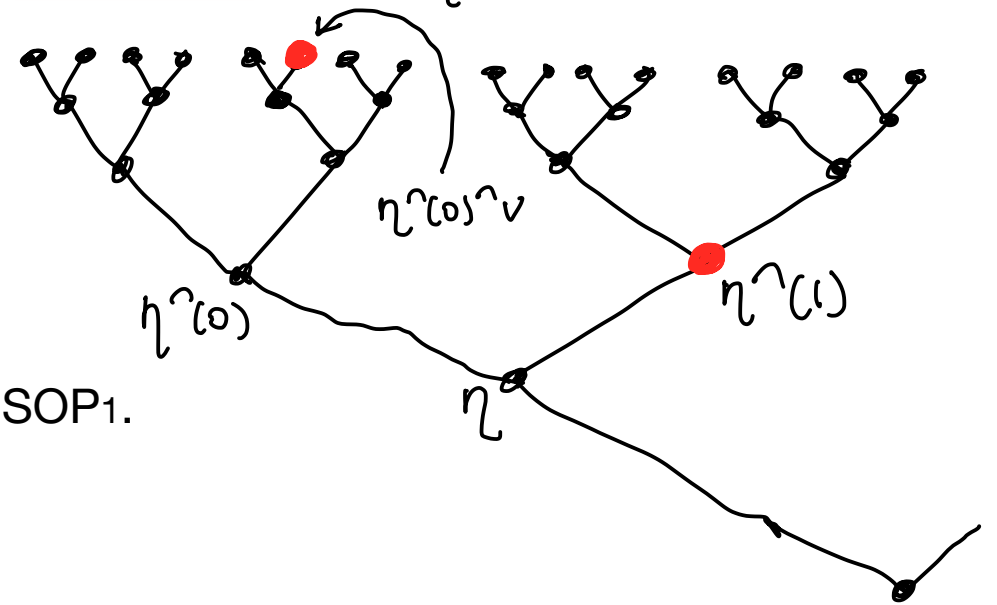
Question. Is there a similar criterion for NSOP_1 theories?

In fact, Kaplan and Ramsey have already found a criterion for NSOP_1 theories and using this, a lot of NSOP_1 theories have been classified. But their criterion works over models, so we want to know if we can extend this to the set-level.

Def. We say T is NSOP1 if there is no $\varphi(x,y) \in L$ and a tree indexed set of parameters $(b_\eta)_{\eta \in 2^{<\omega}}$ such that

(i) $\{\varphi(x,b_\eta) : \eta \lhd \nu\}$ is consistent for all $\nu \in \omega^\omega$,

(ii) $\{\varphi(x,b_{\eta^\frown\langle 0\rangle^\frown\nu}), \varphi(x,b_{\eta^\frown\langle 1\rangle})\}$ is inconsistent for any $\eta, \nu \in \omega^{<\omega}$.



Note. If T is simple, then it is NSOP1.

Def. Let $\downarrow$ be a pre-independence relation and B a set.

(i) A sequence $(a_i)_{i\in\omega}$ is said to be a $\downarrow$-Morley sequence over B if it is indiscernible over B and $a_i \downarrow_B a_{<i}$ for all $i\in\omega$, where $a_{<i}=\{a_j : j<i\}$.

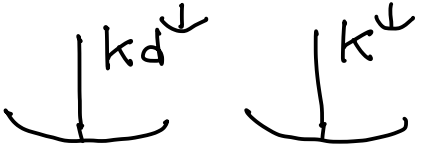
(ii) We say $\varphi(x,a)$ $\downarrow$-Kim-divides over B if there is a $\downarrow$-Morley sequence $(a_i)_{i\in\omega}$ over B with $a_0=a$ such that $\{\varphi(x,a_i)\}_{i\in\omega}$ is inconsistent.

(iii) We say $\varphi(x,a)$ $\downarrow$-Kim-forks over B if it implies a finite disjunction of $\downarrow$-Kim dividing formulas over B.

(iv) We write $a \overset{kd\downarrow}{\underset{C}{\downarrow}} b$ if $tp(a/cb)$ has no $\downarrow$-Kim-dividing formula over C.

(v) We write $a \overset{k\downarrow}{\underset{C}{\downarrow}} b$ if $tp(a/cb)$ has no $\downarrow$-Kim-forking formula over C.

$\overset{kd\downarrow}{\downarrow}$ and $\overset{k\downarrow}{\downarrow}$ are induced pre-independence relations from $\downarrow$.



Def. A global type p(x) is said to be invariant over a set A if

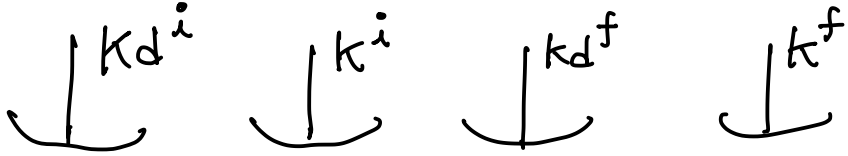
$\varphi(x,a) \leftrightarrow \varphi(x,a') \in p(x)$ for all $\varphi(x,y) \in L$ and $a \equiv_A a'$.

We write $a \downarrow^i_C b$ if there is a global invariant type (over C) containing $tp(a/Cb)$.

Now we can consider $\downarrow^i$-Morley sequences, $\downarrow^i$-Kim-dividing/forking, $\downarrow^{Kd^i}$, $\downarrow^{K^i}$.

We write $\downarrow^{Kd^i}$ for $\downarrow^{Kd^i}$, $\downarrow^{K^i}$ for $\downarrow^{K^i}$. Similarly we write $\downarrow^{Kd^f}$, $\downarrow^{K^f}$ for $\downarrow^{Kd^f}$, $\downarrow^{K^f}$.

Fact. [Kaplan, Ramsey] If T is NSOP1, then $\downarrow^{K^i}$ satisfies monotonicity, existence, right extension, strong finite character, symmetry over models, and the independence theorem over models. Conversely, if there is a pre-independence relation satisfying above properties, then T is NSOP1.



We want to find a pre-independence relation $\downarrow$ satisfying

(i) $\downarrow \approx \downarrow^{f}$ over sets in simple theories,

(ii) $\downarrow = \downarrow^{K^{i}}$ over models in NSOP$_1$ theories,

(iii) Over sets, and in NSOP$_1$, $\downarrow$ satisfies all properties that appear in Kaplan and Ramsey's theorem.

Note. If $\downarrow$ is stronger than $\downarrow^{K^{f}}$ and weaker than $\downarrow^{f}$ (i.e. $\downarrow^{f} \to \downarrow \to \downarrow^{K^{f}}$), then $\downarrow = \downarrow^{f}$ over sets in simple theories. If $\downarrow$ is stronger than $\downarrow^{K^{i}}$ and weaker than $\downarrow^{K^{f}}$ ($\downarrow^{K^{f}} \to \downarrow \to \downarrow^{K^{i}}$), then $\downarrow = \downarrow^{K^{i}}$ over models in NSOP$_1$ theories.

Note. Among the six properties in Kaplan and Ramsey's theorem, monotonicity and right extension are quite trivial. They are relatively easy to obtain. For example, $\downarrow^{K^*}$ always satisfies monotonicity and right extension for any pre-independence relation $\downarrow$.

Fact. Suppose that $\downarrow$ satisfies monotonicity, right extension, and existence. Then for any a and B, there is a $\downarrow$-Morley sequence $(a_i)_{i\in\omega}$ over B with $a_0 = a$.

For some technical reason, the absence of $\downarrow$-Morley sequence yields some inconveniences. For example, without the existence of $\downarrow$-Morley sequence, we can not guarantee that $\downarrow^{K^*} \to \downarrow^{C^*}$, where $\downarrow^{C^*}$ is $\downarrow$-Conant independence.

So first, we want to obtain existence.

Fact. [Hanson] In any theory, $\downarrow^{K^i}$ satisfies existence over sets.

Theorem. [Kim, K., Lee] In NSOP1 theories, $\downarrow^{K^f}$ satisfies existence over sets.

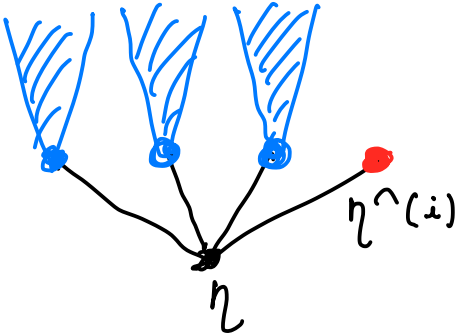
More generally,

Theorem. [Kim, K., Lee] Let T be NSOP1. Suppose $\downarrow$ satisfies monotonicity, right extension, strong finite character, and base monotonicity over sets. If $\downarrow$ is stronger than $\downarrow^{Ku}$ over models, then $\downarrow^{K^\perp}$ satisfies existence, where $\downarrow^{Ku}$ is $\downarrow^u$-Kim-independence.

Note. It is easy to see that $\downarrow^f$ is the weakest pre-independence relation satisfying the above conditions. So $\downarrow^{K^f}$ is the strongest pre- independence relation we can obtain using the same technique.

Corollary. In any NSOP1 theory, any A and B, there is a $\downarrow^{K^f}$-Morley sequence $(a_i)_{i\in\omega}$ over B with $a_0 = a$.

Sketch of proof. Suppose $a \not\downarrow^{K^f}_{\emptyset} \emptyset$. Then there are $\downarrow^f$-Kim-dividing formulas $\psi_0(x, b_0), \cdots, \psi_n(x, b_n)$ such that $tp(a) \vdash \bigvee \psi_i(x, b_i)$. By right extension of $\downarrow^f$ and pigeon hole principle, we can construct an ill-founded tree indexed set of parameters $(b_\eta)_{\eta \in \mathcal{T}_\omega}$ and $\psi(x, y)$ such that



(i) $\psi(x, y)$ and $(b_\eta)_{\eta \in \mathcal{T}_\omega}$ forms a witness of tree property,

(ii) for each η, $(b_{\eta^\frown(i)})_{i \in \omega}$ forms a $\downarrow^f$-Morley sequence,

(iii) for each η, $b_{\eta^\frown(i)} \downarrow^f \{b_\nu \mid \nu \trianglerighteq \eta^\frown(j) \text{ for some } j < i\}$.



Then by base monotonicity and right extension, we can find a model M such that each direct sibling forms a $\downarrow^f$-Morley sequence over M. So we can apply Kim's lemma over models in NSOP1, proved by Kaplan and Ramsey.

$\downarrow^{k^f}$ is the strongest pre-independence relation satisfying monotonicity, right extension, and existence in NSOP$_1$ theories we know so far. In fact, many researchers thought that $\downarrow^{f}$ would have existence in NSOP$_1$, but recently, a quite suprising example was observed.

Fact. [Mutchnik] There is an NSOP$_1$ theory such that $\downarrow^{f}$ does not satisfies existence.

So, for now, our result is the strongest one.

Discussion for further study

The next goal will be finding a pre-independence relation satisfying symmetry and the independence theorem. The first candidate we consider is $\downarrow^{Kf}$.

And we have observed that if $\downarrow^{f}$ and $\downarrow^{Kf}$ satisfies certain conditions, then there is a pre-independence relation satisfying monotonicity, right extension, existence, symmetry, strong finite character, and the independence theorem.

Conjecture 1. Let T be NSOP1. If $\varphi(x,a) \downarrow^{f}$-Kim-divides over B, then $\{\varphi(x,a_i)\}_{i\in\omega}$ is inconsistent for all indiscernible sequence $(a_i)_{i\in\omega}$ over B with $a_0 \equiv a$ such that $a_{<i} \downarrow^{Kf}_{B} a_i$ for all $i\in\omega$.

Observation. If Conjecture 1 is true, then $\downarrow^{\sigma}$ satisfies monotonicity, right extension, existence, symmetry, strong finite character over sets, and the independence theorem in NSOP1, where

$$\downarrow^{\sigma} = \downarrow^{re\left(\downarrow^{ec(\downarrow^{kf})}\right)},$$

where $\downarrow^{re\downarrow}$ is defined by $a \underset{C}{\downarrow^{re\downarrow}} b$ iff $b \underset{C}{\downarrow} a$,

and $\downarrow^{ec\downarrow}$ is the weakest pre-independence relation stronger than $\downarrow$, satisfying right extension and left extension.

It is known that if an NSOP$_1$ theory T satisfies existence axiom (i.e. $\perp^f$ satisfies existence), then Conjecture 1 is true.

Question. Is Conjecture 1 true in any NSOP$_1$ theory?

Thank you