A characterization of treeless theories using collapsing indiscernibles

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RIMS Model Theory Workshop 2025

Indiscernible sequences

Definition

Let T be a complete theory, $\mathbb{M} \models T$ a monster model, $n < \omega$, and A a small subset of $\mathbb{M}$. A sequence $(a_i)_{i \in \omega}$ in $\mathbb{M}^n$ is *indiscernible* over A if for all $m < \omega$ and all $i_0 < \cdots < i_{m-1}$, $j_0 < \cdots < j_{m-1}$ in ω, we have

$$\mathrm{tp}(a_{i_0}, \ldots, a_{i_{m-1}}/A) = \mathrm{tp}(a_{j_0}, \ldots, a_{j_{m-1}}/A).$$

When $A = \emptyset$ we simply say that $(a_i)_{i \in \omega}$ is indiscernible.

More generally, for an infinite linear order $\mathcal{I}$, a sequence $(a_i)_{i \in \mathcal{I}}$ is *indiscernible* over A if for all $m < \omega$ and all $i_0 < \cdots < i_{m-1}$, $j_0 < \cdots < j_{m-1}$ in $\mathcal{I}$, we have

$$\mathrm{tp}(a_{i_0}, \ldots, a_{i_{m-1}}/A) = \mathrm{tp}(a_{j_0}, \ldots, a_{j_{m-1}}/A).$$

Generalized indiscernible sequences

Definition

Let $\mathcal{L}^*$ be a language and $\mathcal{I}^*$ an $\mathcal{L}^*$-structure. Let $\mathcal{L}$ be another language, T an $\mathcal{L}$-theory, $\mathbb{M}$ a monster model of T, and A a small subset of $\mathbb{M}$.

A family $(a_\eta)_{\eta \in \mathcal{I}^*}$ in $\mathbb{M}$ is said to be *$\mathcal{I}^*$-indiscernible* over A if for all finite tuples $\bar{\eta}, \bar{\nu}$ in $\mathcal{I}^*$ with $\mathrm{qftp}_{\mathcal{L}^*}(\bar{\eta}) = \mathrm{qftp}_{\mathcal{L}^*}(\bar{\nu})$, we have

$$\mathrm{tp}_{\mathcal{L}}(a_{\bar{\eta}}) = \mathrm{tp}_{\mathcal{L}}(a_{\bar{\nu}}).$$

Ordinary indiscernible sequences correspond to the case that $\mathcal{I}^*$ is a linear order and $\mathcal{L}^* = \{<\}$.

Tree indiscernible

Let $\mathcal{L}_{\mathrm{str}} := \{\trianglelefteq, \wedge, <_{lex}\}$ and $\mathcal{I}_{\mathrm{str}} = (\omega^{<\omega}; \trianglelefteq^{\mathcal{I}_{\mathrm{str}}}, \wedge^{\mathcal{I}_{\mathrm{str}}}, <_{lex}^{\mathcal{I}_{\mathrm{str}}})$ an $\mathcal{L}_{\mathrm{str}}$-structure such that

- $\eta \trianglelefteq \nu \Leftrightarrow \eta \subseteq \nu$,
- $\eta \wedge \nu := \eta|_{\min\{n<\omega : \eta(n) \neq \nu(n)\}}$,
- $\eta <_{lex} \nu$ if
  - $\eta \subseteq \nu$ or
  - $\nu \not\subseteq \eta$ and $\eta(\min\{n < \omega : \eta(n) \neq \nu(n)\}) < \nu(\min\{n < \omega : \eta(n) \neq \nu(n)\})$.

We say $(a_\eta)_{\eta \in \omega^{<\omega}}$ is *strongly indiscernible* if it is $\mathcal{I}_{\mathrm{str}}$-indiscernible.

Collapsing indiscernibles

Let $\mathcal{L}_0$ be a language and $\mathcal{I}_0$ an $\mathcal{L}_0$-structure. Let $\mathcal{L}_1 \subsetneq \mathcal{L}_0$ and $\mathcal{I}_1 := \mathcal{I}_0|_{\mathcal{L}_1}$. Let I be the universe of $\mathcal{I}_0$ ($\mathcal{I}_1$). If $(a_\eta)_{\eta \in I}$ is $\mathcal{I}_1$-indiscernible, then it is $\mathcal{I}_0$-indiscernible.

Definition

Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. From the same notation above, we say T *collapses $\mathcal{I}_0$-indiscernible to $\mathcal{I}_1$* if every $\mathcal{I}_0$-indiscernible sequence is $\mathcal{I}_1$-indiscernible.

Example

Let $\mathcal{L}_0 := \{<\}$, $\mathcal{L}_1 := \{\}$, and $\mathcal{I}_0 := (\omega, <)$. Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Then the following are equivalent.

- T is stable.
- T collapses $\mathcal{I}_0$-indiscernible to $\mathcal{I}_1$.

Dividing lines characterized by collapsing indiscernibles

Example [Scow]

Let $\mathcal{L}_0 := \{<, R\}$ where R is a binary relation symbol, $\mathcal{L}_1 := \{<\}$, and $\mathcal{I}_0$ be the random ordered graph. Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Then the following are equivalent.

- T is NIP.
- T collapses $\mathcal{I}_0$-indiscernible to $\mathcal{I}_1$.

More generally,

Example [Chernikov, Palacin, Takeuchi]

n-dependence: A theory T is n-dependent if and only if every random ordered $(n+1)$-ary hypergraph indiscernible is an indiscernible sequence.

Dividing lines characterized by collapsing indiscernibles

Example [Guingona, Hill, Scow]

Let $\mathcal{L}_{\text{fn}}$ be a two-sorted language with sorts X_0, X_1, two binary relation symbols $<_0$ in X_0 and $<_1$ in X_1, and a ternary relation symbol $E \subseteq X_0 \times X_1 \times X_1$. Let $\mathcal{L}_2 := \{<_0, <_1\} (\subseteq \mathcal{L}_{\text{fn}} = \{<_0, <_1, E\})$. Let $\mathcal{I}_{\text{fn}}$ be the Fraïssé limit of the class of finite $\mathcal{L}_{\text{fn}}$-structure such that

- $E(x, \cdot, \cdot)$ is an equivalent relation on X_1 for each $x \in X_0$,
- $y = y'$ in X_1 if and only if $\models E(x, y, y')$ for all $x \in X_0$,
- $<_0, <_1$ are linear order in X_0, X_1, respectively,
- for all $n < \omega$, $x^0 <_0 \cdots <_0 x^n$, and $y \in X_1$, $y/\bigcap_{i \leq n} E_{x_i}$ is a $<_1$-convex subset of $y/\bigcap_{i<n} E_{x_i}$, where E_{x_i} is the equivalent relation on X_1 induced by x_i.

Assume that T eliminate imaginary. Then the following are equivalent.

- T is rosy.
- T collapses $\mathcal{I}_{\text{fn}}$-indiscernible to $\mathcal{I}_2$.

Mekler's construction

Definition

A graph is said to be *nice* if it has neither 3-cycles nor 4-cycles, it has at least two vertices, and for any distinct two vertices a, b, there is c such that $R(a, c)$, $\neg R(b, c)$, and $a \neq c \neq b$.

It is known that for every first-order structure M, there exists a nice graph C such that M and C are bi-interpretable.

Definition

For a nice graph $C = (c_i)_{i<\omega}$ and a prime number p, the *Mekler group* of C, $G(C)$ is given by $(X \times Y, *)/N$, where

- $X := \oplus_{i<\omega} c_i \mathbb{Z}_p$, $Y := \oplus_{i<j<\omega} [c_i, c_j]\mathbb{Z}_p$,
- $(\sum_{i<\omega} \alpha_i c_i, d) * (\sum_{i<\omega} \beta_i c_i, e) =$
  $(\sum_{i<\omega} (\alpha_i + \beta_i) c_i, d + e - \sum_{i<j<\omega} \beta_i \alpha_j [c_i, c_j])$,
- $N := \langle \{(0, [c_i, c_j]) : C \models R(c_i, c_j)\} \rangle$.

It is also known that $G(C)$ is a 2-nilpotent group with exponent p and C is interpretable in $G(C)$, but not the converse in general.

Mekler's construction

Fact [Mekler],[Baudish],[Chernikov, Hempel],[Ahn, K., Lee]

- T is stable if and only if $\mathrm{Th}(G(C(T)))$ is stable.
- T is CM-trivial if and only if $\mathrm{Th}(G(C(T)))$ is CM-trivial.
- T is simple if and only if $\mathrm{Th}(G(C(T)))$ is simple.
- T is NIP if and only if $\mathrm{Th}(G(C(T)))$ is NIP.
- T is NTP_1 if and only if $\mathrm{Th}(G(C(T)))$ is NTP_1,
- T is NTP_2 if and only if $\mathrm{Th}(G(C(T)))$ is NTP_2,
- T is NATP if and only if $\mathrm{Th}(G(C(T)))$ is NATP.

where $C(T)$ is the nice graph corresponding to T.

Fact [Boissonneau, Papadopoulos, Touchard, 2024]

Let $\mathcal{A}$ be a class of first-order theories characterized by using collapsing indiscernible. Then the following are equivalent.

- $T \in \mathcal{A}$.
- $\mathrm{Th}_{\mathrm{gp}}(G(C(T))) \in \mathcal{A}$.

In particular, T is rosy if and only if $\mathrm{Th}(G(C(T)))$ rosy for any T.

Treeless theoires

Definition [Kaplan, Ramsey, Simon]

Let $\mathcal{L}_{\mathrm{str},P} := \{\trianglelefteq, \wedge, <_{lex}, P\}$ where P is a unary predicate symbol. We give an $\mathcal{L}_{\mathrm{str},P}$-structure $\mathcal{I}_{\mathrm{str},P}$ on $\omega^{\leq\omega}$ by $P = \omega^{\omega}$. The interpretations of the remaining symbols follows the definition of strongly indiscernible sequences above.

We say a theory T is *treeless* if $(a_\eta)_{\eta\in\omega^\omega}$ is $\{<_{lex}\}$-indiscernible over $a_\emptyset$ for any $\mathcal{I}_{\mathrm{str},P}$-indiscernible $(a_\eta)_{\omega^{\leq\omega}}$.

Example

- DLO is treeless.
- Every stable theory is treeless.
- The theory of any ordered abelian group is not treeless.

Fact [Kaplan, Ramsey, Simon]

- Every treeless theory is rosy.
- If a treeless theory is NSOP_1, then it is simple.

GS-independence in treeless theories

Definition

- We say that a partial type π is *ind-definable* over A if for every $\varphi(x;y)$, the set $\{b : \varphi(x;b) \in \pi\}$ is a union of A-definable sets.
- We say a partial type $\pi(x)$ is *generically stable* over A if π is ind-definable over A and the following holds:
  (GS) if $(a_k)_{k<\omega}$ is such that $a_k \models \pi|_{Aa_{<k}}$ for all $k < \omega$ and $\varphi(x;b) \in \pi$, then for all but finitely many values of k, we have $\varphi(a_k;b)$.
- We write $a \mathop{\smile\!\!\!\!\!\!|}^{GS}_{A} b$ if for every partial type $\pi(x)$ generically stable over A, if $b \models \pi|_A$, then $b \models \pi|_{Aa}$.

Fact [Kaplan, Ramsey, Simon]

- In any theory, $\mathop{\smile\!\!\!\!\!\!|}^{GS}$ is weaker than $\mathop{\smile\!\!\!\!\!\!|}^{f}$.
- In any theory, $\mathop{\smile\!\!\!\!\!\!|}^{GS}$ satisfies invariance, normality, monotonicity, existence, extension, finite character, transitivity, local character, a weak form of local character, anti-reflexivity, and algebraicity.
- In treeless theories, $\mathop{\smile\!\!\!\!\!\!|}^{GS}$ satisfies symmetry and base monotonicity.

Definition

We say T is *weakly treeless* if $(a_\eta)_{\eta \in \omega^\omega}$ is $\{<_{lex}\}$-indiscernible for any $\mathcal{I}_{\mathrm{str},P}$-indiscernible $(a_\eta)_{\eta \in \omega^{\leq \omega}}$.

Fact [Kaplan, Ramsey, Simon]

- If T is treeless, then it is weakly treeless.
- If T is NIP and weakly treeless, then it is treeless.

Question [Kaplan, Ramsey, Simon]

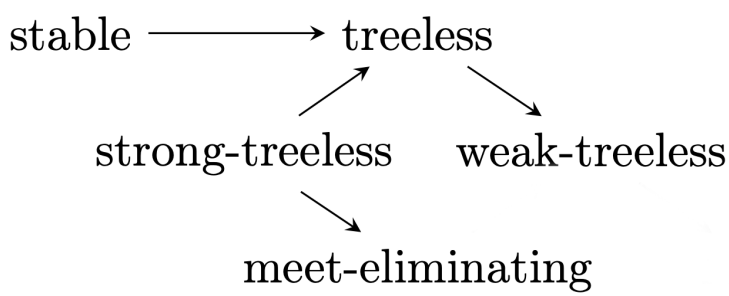
Does weak-treelessness imply treelessness in general?

Definition

- We say T is *strongly treeless* if every $\{\trianglelefteq, \wedge, <_{lex}, P\}$-indiscernible $(a_\eta)_{\eta \in \omega^{\leq \omega}}$ is $\{\trianglelefteq, <_{lex}, P\}$-indiscernible.
- We say T has *meet-elimination* if every $\{\trianglelefteq, \wedge, <_{lex}\}$-indiscernible $(a_\eta)_{\eta \in \omega^{<\omega}}$ is $\{\trianglelefteq, <_{lex}\}$-indiscernible.

Proposition

- If T is strongly treeless, then it is treeless.
- If T is strongly treeless, then it has meet-elimination.



Notation

$\mathcal{L}_\delta := \{\Delta, <_{lex}\}$, where Δ is a quaternary relation symbol. We give an $\mathcal{L}_\delta$-structure on ω^ω by defining

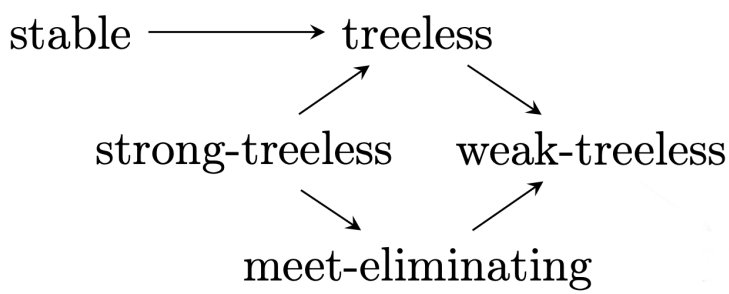
$$\Delta(\eta_0, \eta_1, \eta_2, \eta_3) \text{ if and only if } \eta_0 \wedge \eta_1 \trianglelefteq \eta_2 \wedge \eta_3.$$

Proposition

The following are equivalent

- T is weakly treeless.
- Every $\{\Delta, <_{lex}\}$-indiscernible $(a_\eta)_{\eta \in \omega^\omega}$ is $\{<_{lex}\}$-indiscernible.

In particular, meet-eliminating implies weak-treelessness.



Theorem

If T is treeless, then it is strongly treeless.

$$\text{stable} \to \text{treeless} \to \text{meet-eliminating} \to \text{weak-treeless}$$

Corollary

T is treeless if and only if $\mathrm{Th}(G(C(T)))$ is treeless.

Question

- Is there a meet-eliminating theory which is not treeless?
- Is there a weakly treeless theory which is not meet-eliminating?

Notation

$\mathcal{L}_C := \{C, <_{lex}\}$, where C is a ternary relation symbol. We give an $\mathcal{L}_C$-structure on 2^ω by defining

$$C(\eta_0, \eta_1, \eta_2) \text{ if and only if } \eta_0 \wedge \eta_1 \wedge \eta_2 \lhd \eta_2 \wedge \eta_3.$$

We can give an interpretation of C in ω^ω in a similar way. Then, in fact, C can be described by Δ and vice versa. So they are essentially the same.

Definition

We say T is *C-less* if every $\{C, <_{lex}\}$-indiscernible $(a_\eta)_{\eta \in 2^\omega}$ is $\{<_{lex}\}$-indiscernible.

Proposition

If T is weakly treeless, then it is C-less.

Corollary

If T is treeless, then it is C-less.

$$\text{stable} \to \text{treeless} \to \text{meet-eliminating} \to \text{weak-treeless} \to \mathcal{C}\text{-less}$$

Notation

- We say $X \subseteq 2^{<\omega}$ is an *antichain* if $\eta \not\trianglelefteq \nu$ and $\nu \not\trianglelefteq \eta$ for all distinct $\eta, \nu \in X$.
- We say $X \subseteq 2^{<\omega}$ is a *descending comb* if $\eta \wedge \nu \rhd \nu \wedge \xi$ for all $\eta, \nu, \xi \in X$ with $\eta <_{lex} \nu <_{lex} \xi$.

Definition

- We say T is *NATP* if there do not exist $\varphi(x, y)$ and $(a_\eta)_{\eta \in 2^{<\omega}}$ such that
  - $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for any $\eta \triangleleft \nu$,
  - $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any antichain $X \subseteq 2^{<\omega}$.
- [Mutchnik] We say T is *NCTP* if there do not exist $\varphi(x, y)$, $(a_\eta)_{\eta \in 2^{<\omega}}$, and $k \in \omega$ such that
  - $\{\varphi(x, a_{\eta|_n})\}_{n<\omega}$ is k-inconsistent for any $\eta \in 2^\omega$,
  - $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any descending comb $X \subseteq 2^{<\omega}$.

Fact

- If T is NTP_1 or NTP_2, then it is NCTP.
- If T is NCTP, then it is NATP.

Proposition

If T is C-less and NATP, then it is NCTP.

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