n-variable theorems of NCTP, NBTP, and NSOP_3

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By a *language*, we mean a set $\mathcal{L}$ that consists of three kinds of symbols; predicates, functions, and constants.

Let $\mathcal{L}$ be a language. By an *$\mathcal{L}$-theory*, we mean a consistent set of sentences written in $\mathcal{L}$ with logical symbols $\wedge, \vee, \neg, \rightarrow, \exists, \forall$. By consistent, we mean that the sentences in the set does not yield a contradiction. Once a set of sentences is consistent, then there must exists a mathematical structure where all the sentences are realized simultaneously.

One of main topics of model theory is finding reasonable ways of classifying first-order theories and investigating theories under these classifications. There are many ways of doing this.

- Consistency-inconsistency configuration (NTP, NTP_1, NTP_2),
- Existence of a certain independence notion (stable, simple, NSOP_1, rosy),
- Collapsing indiscernible (stable, NIP, k-NIP, NFOP_2),
- Using rank notions (dp-minimal, SU-rank),
- Restricting shapes of definable sets (o-minimal, C-minimal),

  and more.

In this talk, we focus on the first method.

Consistency-inconsistency configuration

Let $\mathcal{L}_0$ be a language and $\mathcal{I}_0$ an $\mathcal{L}_0$-structure. Let $\mathcal{A}$ and $\mathcal{B}$ be sets of $\mathcal{L}_0$-substructures of $\mathcal{I}_0$ such that $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Consider the class of first-order theories given by

$\{\ T :$ T is an $\mathcal{L}$-theory for some language $\mathcal{L}$,
there exist $\varphi(x, y) \in \mathcal{L}$ and $(a_i)_{i \in \mathcal{I}_0} \subseteq \mathbb{M}$ for some saturated $\mathbb{M} \models T$ such that

- $\{\varphi(x, a_i)\}_{i \in A}$ is consistent for all $A \in \mathcal{A}$,
- $\{\varphi(x, a_i)\}_{i \in B}$ is inconsistent for all $B \in \mathcal{B}$ $\}$.

There are many dividing lines of first-order theories defined in this way.

Examples of consistency-inconsistency configuration

$\omega^{<\omega}$ is the set of functions from a natural number to ω. For $\eta, \nu \in \omega^{<\omega}$, we write $\eta \trianglelefteq \nu$ if $\eta \subseteq \nu$ (functions are sets of ordered pairs). Then $(\omega^{<\omega}, \trianglelefteq)$ forms a poset. A subset X in $\omega^{<\omega}$ is called a *path* if it is linarly ordered by $\trianglelefteq$. X is called a *direct sibling set* if $X = \{\eta^\frown(i)\}_{i \in I}$ for some $\eta : n \to \omega$ and $I \subseteq \omega$, where $\eta^\frown(i) : n+1 \to \omega$ maps n to i and k to $\eta(k)$ for $k < n$.

Example

We say an $\mathcal{L}$-theory T with its saturated model $\mathbb{M}$ has the *tree property (TP)* if there exist $\varphi(x, y) \in \mathcal{L}$, $(a_\eta)_{\eta \in \omega^{<\omega}} \subseteq \mathbb{M}$, and $k < \omega$ such that

- $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any path X,
- $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is k-inconsistent for any direct sibling set X.

We say T has TP_1 if there exist $\varphi(x, y) \in \mathcal{L}$ and $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

- $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any path X,
- $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for any incomparable η, ν.

We say T has TP_2 if there exist $\varphi(x, y) \in \mathcal{L}$, $(a_{i,j})_{i,j<\omega}$, and $k < \omega$ such that

- $\{\varphi(x, a_{i,f(i)})\}_{i<\omega}$ is consistent for any function $f : \omega \to \omega$,
- $\{\varphi(x, a_{i,j}\}_{j<\omega}$ is k-inconsistent for any $i < \omega$.

We say T is *NTP (NTP_1, NTP_2)* if it does not have TP (TP_1, TP_2).

Examples of consistency-inconsistency configuration

Stability and NIP are also important dividing lines in model theory. Recently, it is proved that they can be characterized by consistency-inconsistency configurations as well.

Example [Day]

T is *unstable* if there exist $\varphi(x,y) \in \mathcal{L}$, $n < \omega$, $(a_\eta)_{\eta \in \mathbf{c}_n^{<\omega}}$, and $c_0, ..., c_{k-1} \in \mathbf{c}_n$ such that

- $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any path X,
- $\{\varphi(x, a_{\eta^\frown(c_0')}), ..., \varphi(x, a_{\eta^\frown(c_{k-1}')})\}$ is inconsistent for all $c_0'...c_{k-1}' \equiv c_0...c_{k-1}$,

T has *IP* if there exist $\varphi(x,y) \in \mathcal{L}$, $n < \omega$, $(a_{i,c})_{i<\omega, c \in \mathbf{c}_n}$, and $c_0, ..., c_{k-1} \in \mathbf{c}_n$ such that

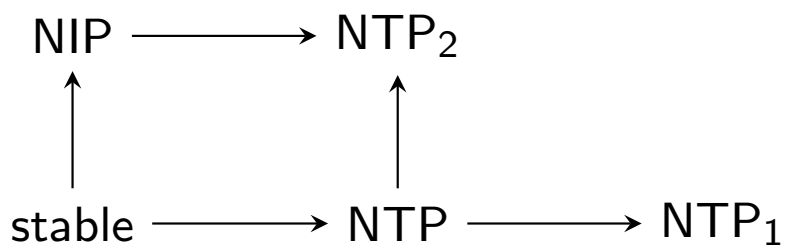
- $\{\varphi(x, a_{i,f(i)})\}_{i<\omega}$ is consistent for any function $f : \omega \to \mathbf{c}_n$,
- $\{\varphi(x, a_{i,c_0'}), ..., \varphi(x, a_{i,c_{k-1}'})\}$ is inconsistent for any $i < \omega$ and $c_0'...c_{k-1}' \equiv c_0...c_{k-1}$,

where $\mathbf{c}_n$ is the Fraïssé limit of n-colored linear order.

We say T is *stable (NIP, resp.)* if it is not unstable (does not have IP, resp.).

Fact [Shelah]

- If a theory has TP_1 or TP_2, then it has TP.
- If a theory has TP, then it has TP_1 or TP_2.
- If a theory has TP_2, then it has IP.
- If a theory has TP or IP, then it is unstable.
- If a theory is unstable, then it has TP or IP.



One-variable theorem

Let $\mathcal{L}_0$ be a language and $\mathcal{I}_0$ an $\mathcal{L}_0$-structure. Let $\mathcal{A}$ and $\mathcal{B}$ be sets of $\mathcal{L}_0$-substructures of $\mathcal{I}_0$ such that $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Let R and NR be properties define as below.

For a language $\mathcal{L}$ and an $\mathcal{L}$-theory T, we say T has R if there exist $\varphi(x, y) \in \mathcal{L}$ and $(a_\eta)_{\eta \in \mathcal{I}_0} \subseteq \mathbb{M} \models T$ such that

- $\{\varphi(x, a_i)\}_{i \in A}$ is consistent for all $A \in \mathcal{A}$,
- $\{\varphi(x, a_i)\}_{i \in B}$ is inconsistent for all $B \in \mathcal{B}$.

We say T is NR if it does not have R.

Then by the one-variable theorem for R (sometime we call it one-variable theorem for NR, without distinguishing), we mean the following statement.

One-variable theorem for R

If T has R, then there exist $\varphi(x, y)$ and $(a_\eta)_{\eta \in \mathcal{I}_0}$ such that

- $\{\varphi(x, a_i)\}_{i \in A}$ is consistent for all $A \in \mathcal{A}$,
- $\{\varphi(x, a_i)\}_{i \in B}$ is inconsistent for all $B \in \mathcal{B}$,
- $|x| = 1$.

Stability, IP, TP_1, and TP_2 satisfy one-variable theorems.

Fact

[Shelah] (combining with Day's result) If T is unstable, then there exist $\varphi(x,y) \in \mathcal{L}$, $n < \omega$, $(a_\eta)_{\eta \in \mathbf{c}_n^{<\omega}}$, and $c_0, \ldots, c_{k-1} \in \mathbf{c}_n$ such that

- $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any path X,
- $\{\varphi(x, a_{\eta^\frown(c_0')}), \ldots, \varphi(x, a_{\eta^\frown(c_{k-1}')})\}$ is inconsistent for all $c_0' \ldots c_{k-1}' \equiv c_0 \ldots c_{k-1}$,
- $|x| = 1$.

[Shelah] (combining with Day's result) If T is IP, then there exist $\varphi(x,y) \in \mathcal{L}$, $n < \omega$, $(a_{i,c})_{i<\omega, c \in \mathbf{c}_n}$, and $c_0, \ldots, c_{k-1} \in \mathbf{c}_n$ such that

- $\{\varphi(x, a_{i,f(i)})\}_{i<\omega}$ is consistent for any function $f : \omega \to \mathbf{c}_n$,
- $\{\varphi(x, a_{i,c_0'}), \ldots, \varphi(x, a_{i,c_{k-1}'})\}$ is inconsistent for all $i < \omega$, $c_0' \ldots c_{k-1}' \equiv c_0 \ldots c_{k-1}$,
- $|x| = 1$.

[Chernikov, Ramsey] If T is TP_1, then there exist $\varphi(x,y)$ and $(a_\eta)_{\eta \in \mathcal{I}_0}$ such that

- $\{\varphi(x, a_i)\}_{i \in A}$ is consistent for all $A \in \mathcal{A}$,
- $\{\varphi(x, a_i)\}_{i \in B}$ is inconsistent for all $B \in \mathcal{B}$,
- $|x| = 1$.

[Chernikov] If T is TP_2, then there exist $\varphi(x,y)$ and $(a_\eta)_{\eta \in \mathcal{I}_0}$ such that

- $\{\varphi(x, a_i)\}_{i \in A}$ is consistent for all $A \in \mathcal{A}$,
- $\{\varphi(x, a_i)\}_{i \in B}$ is inconsistent for all $B \in \mathcal{B}$,
- $|x| = 1$.

- One-variable theorem for a property R can make it easier to check if a theory has R.
- It can be applied on some preservation problems, as below.

Fact [Chernikov]

$\mathrm{Th}(\mathcal{K})$ has TP_2 if and only if $\mathrm{Th}(k)$ has TP_2,
where $\mathcal{K} = (K, \Gamma, k, v : K \to \Gamma, ac : K \to k)$ is a henselian valued field of characteristic $(0, 0)$ in the Denef-Pas language
$\mathcal{L}_{\mathrm{Pas}} := \mathcal{L}_K \cup \mathcal{L}_{\Gamma,\infty} \cup \mathcal{L}_k \cup \{ac, v\}$ (3-sorted language).

Fact [d'Elbée, Kaplan, Neuhauser]

If T is NIP, then ACF_T is NIP,
where T is a theory of field in some language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{ring}}$,
and ACF_T is the $\mathcal{L} \cup \{P\}$-theory of algebraically closed fields M with a distinguished subfield $P(M) \models T$. (P is unary predicate symbol.)

Fact [Berenstein, d'Elbée, Vassiliev]

T is NTP_2 (NTP_1) if and only if T_G is NTP_2 (NTP_1, resp).
where T is a theory of vectos spaces of a field $\mathbb{F}$ with characteristic 0 in some language $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{ring}}$, and T_G is the expansion of T with a dense/codense generic submodule G.

- An $\mathcal{L}$-theory T is said to be *geometric* if it eliminate $\exists^\infty$ and acl satisfies exchange property.
- $\dim(B/C)$ is the maximal cardinality of $B_0 \subseteq B$ such that $b_0 \notin \mathrm{acl}((B_0 \setminus \{b_0\}) \cup C)$ for all $b_0 \in B_0$. If $C = \emptyset$, then we write $\dim(B) := \dim(B/C)$.
- Let T be an $\mathcal{L}$-theory and $\mathbb{M} \models T$. A subset $A \subseteq \mathbb{M}^1$ is said to be *dense/codense* if it satisfies
  - for a non-algebraic type $p(x) \in S_1(B)$ with $\dim(B) < \omega$, there exists $m, m' \models p(x)$ such that $m \in A$ and $m' \notin \mathrm{acl}(AB)$.
- Let T be geometric in a language $\mathcal{L}$ and $\mathbb{M} \models T$. Let H be a new unary predicate symbol. We say an expansion $(\mathbb{M}, H(\mathbb{M}))$ is a *lovely pair* if $H(\mathbb{M})$ is dense/codense and $H(\mathbb{M}) \prec \mathbb{M}$. We say $(\mathbb{M}, H(\mathbb{M}))$ is a *H-structure* if $H(\mathbb{M})$ is dense/codense and algebraically independent $(\dim(H(\mathbb{M})) = |H(\mathbb{M})|)$.
- For a H-structure $(\mathbb{M}, H(\mathbb{M}))$ of T and $\mathcal{L}_R := \{R_{\varphi(\bar{x})}(\bar{x}) : \varphi \in \mathcal{L}\}$, the $\mathcal{L}_R$ structure on $H(\mathbb{M})$ is called the *generic trivialization* of $\mathbb{M}$.
- By T_P, T^{ind}, and T^{gt}, we mean the theory of lovely pairs, H-structures, and generic trivializations of T, respectrly.

Many preservation results on lovely pairs, H-structures, and generic trivializations have been established. The following is the list of the preservations results whose proof contains one-variable theorem.

Fact

- [Berenstein, Vassiliev] T is NIP if and only if T^{gt} is NIP.
- [Berenstein, Kim] T is NTP_2 if and only if T_P is NTP_2, and if and only if T^{ind} is NTP_2.
- [Dobrowolski, Kim] T is NTP_1 if and only if T_P is NTP_1, and if and only if T^{ind} is NTP_1.

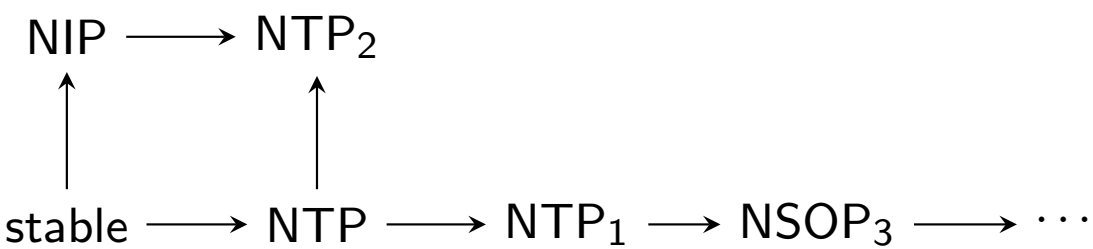
Consistency-inconsistency configurations that we do not know whether the one-variable theorem holds yet

Let $n > 2$. We say T has SOP_n if there exist $\varphi(x, y)$ with $|x| = |y|$ and $(a_i)_{i<\omega}$ such that

- $\models \varphi(a_i, a_j)$ for all $i < j$,
- $\models \neg\exists x_0, ..., x_{n-1}(\varphi(x_0, x_1) \wedge \cdots \wedge \varphi(x_{n-2}, x_{n-1}) \wedge \varphi(x_{n-1}, x_0)$.

Fact

- [Shelah] If T has SOP_{n+1}, then it has SOP_n.
- [Shelah, Džamonja] If T has SOP_3, then it has SOP_2.
- T has SOP_2 if and only if it has TP_1.



Consistency-inconsistency configurations that we do not know whether the one-variable theorem holds yet

Fact [Shelah]

T has SOP_3 if and only if there exist $\varphi(x, y)$ and $(a_{i,j})_{i<\omega, j<2}$ such that

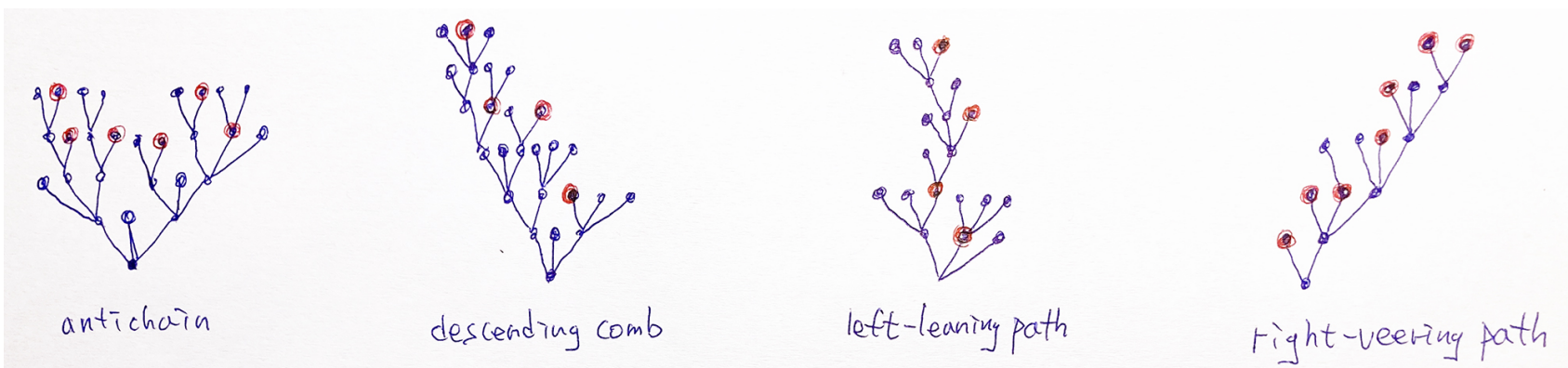
- $\{\varphi(x, a_{i,0})\}_{i<j} \cup \{\varphi(x, a_{i,1})\}_{i>j}$ is consistent for each $j < \omega$,
- $\{\varphi(x, a_{i,1}), \varphi(x, a_{j,0})\}$ is inconsistent for all $i < j$.

SOP_3 can be defined by using consistency-inconsistency configuration, but we still do not know whether the one-variable theorem for SOP_3 is true or not.

Consistency-inconsistency configurations that we do not know whether the one-variable theorem holds yet

Recently, as attempts to find common generalized dividing lines of NTP_1 and NTP_2, a few new consistency-inconsistency configurations have been introduced.

- $X \subseteq \omega^{<\omega}$ is an *antichain* if $\eta \not\trianglelefteq \nu \not\trianglelefteq \eta$ for all $\eta, \nu \in X$.
- $X := \{\eta_0, ...\} \subseteq \omega^{<\omega}$ is a *descending comb* if it is an antichain, $\eta_i >_{lex} \eta_j$ for all $i < j$, and $\eta_i \wedge \eta_k \trianglelefteq \eta_i \wedge \eta_k$ for all $i < j < k$.
- $X := \{\eta_0, ...\} \subseteq \omega^{<\omega}$ is a *left-leaning path* if for all $n < \omega$, there exists ν_n and $i \leq j$ such that $\nu_n^\frown(i) \trianglelefteq \eta_{n+1}$ and $\nu_n^\frown(j) = \eta_n$.
- $X := \{\eta_0, ...\} \subseteq \omega^{<\omega}$ is a *right-veering path* if for all $n < \omega$, there exists ν_n and $i < j$ such that $\nu_n^\frown(i) = \eta_n$ and $\nu_n^\frown(i) \trianglelefteq \eta_{n+1}$.
- $X := \{\eta_0, ...\} \subseteq \omega^{<\omega}$ is a *strict left-leaning path* if for all $n < \omega$, there exists ν_n and $i < j$ such that $\nu_n^\frown(i) \trianglelefteq \eta_{n+1}$ and $\nu_n^\frown(j) = \eta_n$.



Consistency-inconsistency configurations that we do not know whether the one-variable theorem holds yet

Definition

- [Ahn, K.] We say T has ATP if there exist $\varphi(x, y)$ and $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  - $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any antichain X,
  - $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is inconsistent for any path X.
- [Mutchnik] We say T has CTP if there exist $\varphi(x, y)$ and $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  - $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any descending comb X,
  - $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is inconsistent for any path X.
- [Kruckman, Ramsey] We say T has BTP if there exist $\varphi(x, y)$ and $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  - $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any left-leaning path X,
  - $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is inconsistent for any right-veering path X.

Remark

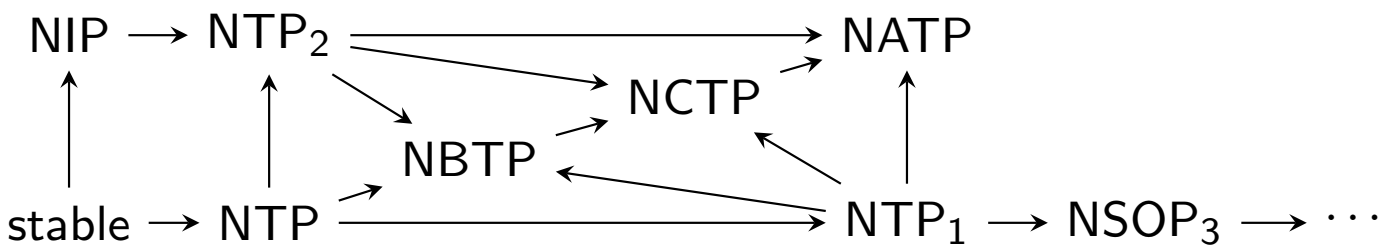
T has BTP if and only if there exist $\varphi(x, y)$ and $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

- $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any strict left-leaning path X,
- $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is inconsistent for any right-veering path X.

Consistency-inconsistency configurations that we do not know whether the one-variable theorem holds yet

Fact

- [Kruckman, Ramsey] If T has NTP_1 or NTP_2, then it is NBTP.
- [Kruckman, Ramsey], [Hanson] If T has NBTP, then it is NCTP.
- [Mutchnik] If T has NCTP, then it is NATP.



Consistency-inconsistency configurations that we do not know whether the one-variable theorem holds yet

Fact [Ahn, K., Lee]

If T has ATP, then there exist $\varphi(x, y)$ and $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

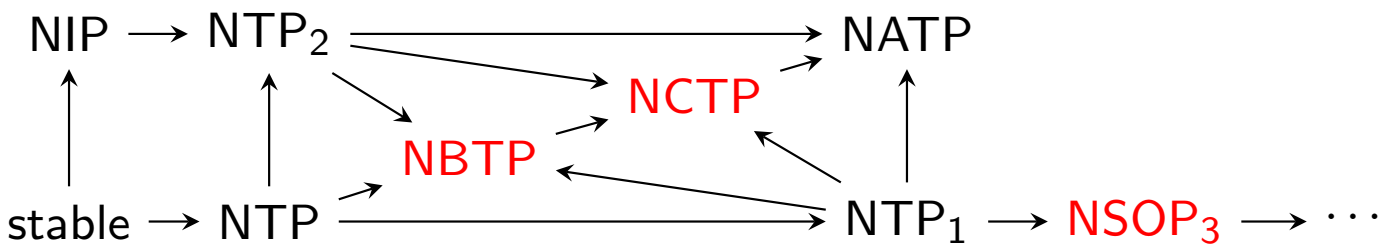
- $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any antichain X,
- $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is inconsistent for any path X,
- $|x| = 1$.

Fact

- [Ahn, K., Lee] $\mathrm{Th}(\mathcal{K})$ has NATP if and only if $\mathrm{Th}(k)$ has NATP.
- [Ahn, K., Lee, Lee] If T is NATP, then ACF_T is NATP,
- [Ahn, K., Lee, Lee] T is NATP if and only if T^{ind} is NATP, and if and only if T_P is NATP.
- [Ahn, K., Lee, Lee] T is NATP if and only if T_G is NATP.
- [K.] T is NATP if and only if T^{gt} is NATP.

But we still do not know whether one-variable theorems for CTP and BTP are true or not.

Consistency-inconsistency configurations that we do not know whether the one-variable theorem holds yet



n-variable theorems

Recall the definition of algebraic dimension. For a tuple $\bar{b} := (b_0, ..., b_{n-1})$ and a set A, $\dim(\bar{b}/A)$ is the maximal cardinality of subtuples $\bar{b}' \subseteq \bar{b}$ such that $b'_i \notin \mathrm{acl}((\bar{b}' \setminus \{b'_i\}) \cup A)$ for all $b'_i \in \bar{b}'$. For a type $\Phi(\bar{x})$ over A is given, then we put $\dim(\Phi) = \max\{\dim(\bar{b}/A) : \bar{b} \models \Phi(\bar{x})\}$.

Theorem [K.]

- If T has CTP, then there exist $\varphi(x, y)$ and $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  - $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any descending comb X,
  - $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is inconsistent for any path X,
  - $\dim(\{\varphi(x, a_\eta)\}_{\eta \in X}) = |x|$ for any descending comb X.
- If T has BTP, then there exist $\varphi(x, y)$ and $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  - $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any strict left-leaning path X,
  - $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is inconsistent for any right-veering path X,
  - $\dim(\{\varphi(x, a_\eta)\}_{\eta \in X}) = |x|$ for any strict-left leaning path X.
- Assume that acl = dcl in T. If T has SOP_3, then there exist $\varphi(x, y)$ and $(a_{i,j})_{i<\omega, j<2}$ such that
  - $\{\varphi(x, a_{i',0})\}_{i'<i} \cup \{\varphi(x, a_{i',1})\}_{i'>i}$ is consistent for any $i < \omega$,
  - $\{\varphi(x, a_{i,1}), \varphi(x, a_{i',0})\}$ is inconsistent for any $i < i'$,
  - $\dim(\{\varphi(x, a_{i',0})\}_{i'<i} \cup \{\varphi(x, a_{i',1})\}_{i'>i}) = |x|$ for any $i < \omega$.

We will call statements in this form n-variable theorems.

n-variable theorems

n-variable theorems can be seen as weakened version of one-variable theorems.

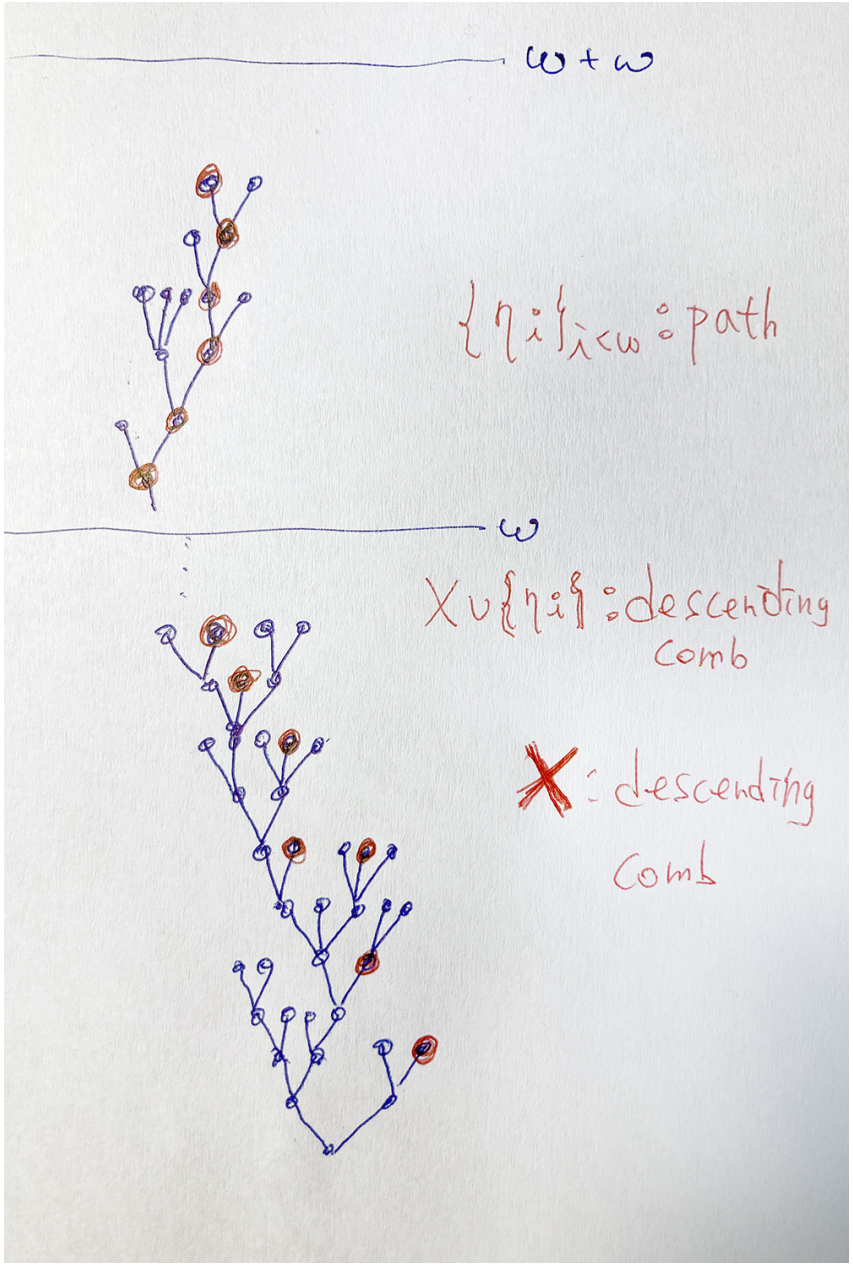
Remark

If there exists $\varphi(x, y)$ with $|x| = 1$ witnessing CTP, then $\dim(\{\varphi(x, a_\eta)\}_{\eta \in X} = 1 = |x|$ for any descending comb X.

why Suppose $\varphi(x, y)$ witnesses CTP with $(a_\eta)_{\eta \in \omega^{<\omega}}$. By compactness we can extend $(a_\eta)_{\eta \in \omega^{<\omega}}$ to $(a_\eta)_{\eta \in \omega^{<\omega+\omega}}$. Choose any descending comb X in $\omega^{<\omega}$. We can find a path $\{\eta_i\}_{i<\omega} \subseteq \omega^{<\omega+\omega} \setminus \omega^{<\omega}$ such that $X_i := X \cup \{\eta_i\}$ is a descending comb for each $i < \omega$. Let $b_i \models \{\varphi(x, a_\eta)\}_{\eta \in X_i}$. Since $\{\eta_i\}_{i<\omega}$ is a path, we may assume $b_i \neq b_j$ for all $i \neq j$. Since $b_i \models \{\varphi(x, a_\eta)\}_{\eta \in X}$ for all $i < \omega$, the type $\{\varphi(x, a_\eta)\}_{\eta \in X}$ has infinitely many solution. Thus $\dim(\{\varphi(x, a_\eta)\}_{\eta \in X} = 1 = |x|$.

Thus if CTP satisfies one-variable theorem, then it satisfies n-variable theorem. The same argument works on BTP and SOP_3 (and stable, NIP, NTP, NTP_1, NTP_2, and ATP as well).

n-variable theorems



n-variable theorems

Good things of n-variable theorems

- It is much easier to prove than one-variable theorem. The argument is quite uniform.
- It is still useful, especially when the algebraic dimension is well-behaving. (ex. geometric theories)

Theorem [K.]

Let T be a geometric theory. Then T is NCTP (NBTP) if and only if T^{gt} is NCTP (NBTP), if and only if T_P is NCTP (NBTP), and if and only if T^{ind} is NCTP (NBTP, resp).
The similar result holds for NSOP_3 when acl = dcl.

Proofs

Let $|x| = 1$. Let $\Gamma(x; y) := \{\sigma(x; y) \to \exists^n x(x; y) : n < \omega, \sigma \in \mathcal{L}\}$.
Then $(a, b) \models \Gamma(x; y)$ if and only if $a \notin \mathrm{acl}(b)$.
For $\bar{x} := (x_0, ..., x_{n-1})$, we put $\check{x}^i := (x_0, ..., x_{i-1}, x_{i+1}, ..., x_{n-1})$.

Proof of n-variable theorem for CTP

Suppose $|\bar{x}| = n$ and $\varphi(\bar{x}, y)$ witnesses CTP with $(a_\eta)_{\eta \in \omega^{<\omega}}$. We may assume $(a_\eta)_{\eta \in \omega^{<\omega}}$ is strongly indiscernible. Consider the descending comb $\{0^i{}^\frown(1)\}_{i<\omega}$. If the type

* $\Phi(\bar{x}, (a_{0^i{}^\frown(1)})_{i<\omega}) := \{\varphi(\bar{x}, a_{0^i{}^\frown(1)})\}_{i<\omega} \cup \Gamma(\bar{x}, (a_{0^i{}^\frown(1)})_{i<\omega})$

is consistent, then $\dim(\{\varphi(\bar{x}, a_{0^i{}^\frown(1)})\}_{i<\omega}) = |x|$. Suppose not. Since φ witnesses CTP, we have $\bar{b} \models \{\varphi(\bar{x}, a_{0^i{}^\frown(1)})\}_{i<\omega}$. Since Φ is not consistent, there exist $i < n$ and $\sigma(x_i; \check{x}^i, \bar{y}) \in \mathcal{L}$ such that

* $\bar{b} \models \sigma(x_i; \check{x}^i, a_{0^0{}^\frown(1)}, ..., a_{0^{m-1}{}^\frown(1)}) \wedge \exists^{<k} x \sigma(x, \check{x}^i, a_{0^0{}^\frown(1)}, ..., a_{0^{m-1}{}^\frown(1)})$.

Let

* $\varphi'(\check{x}^i, y, \bar{y}) := \exists x_i(\varphi(\bar{x}, y) \wedge \sigma(x_i; \check{x}^i, \bar{y}) \wedge \exists^{<k} x \sigma(x, \check{x}^i, \bar{y}))$,
* $a^*_\eta := (a_{0^m{}^\frown\eta}, \bar{a}_{0^{<m}{}^\frown(1)}) = (a_{0^m{}^\frown\eta}, a_{0^0{}^\frown(1)}, ..., a_{0^{m-1}{}^\frown(1)})$.

Then $\varphi'(\check{x}^i, y, \bar{y})$ witness CTP with $(a^*_\eta)_{\eta \in \omega^{<\omega}}$. We can repeat this process until we obtain a witness satisfying n-variable theorem (recall that if $|x| = 1$, then the witness always satisfies n-variable theorem).

Proofs

Consistency of $\{\varphi'(\check{x}^i, a^*_\eta)\}_{\eta\in X}$ for descending combs is easy. The inconsistency case is also easy, but we leave some technical remark on that to explain why we need acl = dcl in the SOP_3 cases.

Why is $\{\varphi'(\check{x}^i, a^*_\eta)\}_{\eta\in X}$ inconsistent for any path X?

Fix a path X and suppose $\check{b}^i \models \{\varphi'(\check{x}^i, a^*_\eta)\}_{\eta\in X}$. Then

* $\check{b}^i \models \exists x_i(\varphi(\bar{x}, a_{0^m\frown\eta}) \wedge \sigma(x_i; \check{x}^i, \bar{a}_{0<m\frown(1)}) \wedge \exists^{<k} x\sigma(x, \check{x}^i, \bar{a}_{0<m\frown(1)})$

for each $\eta \in X$. Thus for each $\eta \in X$, there exists b^η_i such that

* $(b^\eta_i, \check{b}^i) \models \varphi(\bar{x}, a_{0^m\frown\eta}) \wedge \sigma(x_i; \check{x}^i, \bar{a}_{0<m\frown(1)}) \wedge \exists^{<k} x\sigma(x, \check{x}^i, \bar{a}_{0<m\frown(1)})$,

in particular

* $(b^\eta_i, \check{b}^i) \models \varphi(\bar{x}, a_{0^m\frown\eta})$,
* $b^\eta_i \models \sigma(x_i; \check{b}^i, \bar{a}_{0<m\frown(1)}) \wedge \exists^{<k} x\sigma(x, \check{b}^i, \bar{a}_{0<m\frown(1)})$.

Since φ witness CTP with $(a_\eta)_{\eta\in\omega^{<\omega}}$ and $|X| = \omega$, we may assume $b^\eta_i \neq b^\nu_i$ for all $\eta \neq \nu$. But $b^\eta_i \in \mathrm{acl}(\check{b}^i \bar{a}_{0<m\frown(1)})$. It is a contradiction.

The condition $|X| = \omega$ does not hold in the SOP_3 cases.
acl = dcl is assumed to solve this problem.

Proofs

Proof of n-variable theorem for SOP_3

Suppose $|\bar{x}| = n$ and $\varphi(\bar{x}, y)$ witnesses SOP_3 with $(a_{q,j})_{q\in\mathbb{Q},j<2}$. We may assume $(a_{q,0}a_{q,1})_{q\in\mathbb{Q}}$ is order-indiscernible. Consider $X_0 := \{(q,0)\}_{q<0} \cup \{(q,1)\}_{q>0}$. If the type

* $\Phi(\bar{x}, (a_\eta)_{\eta\in X_0}) := \{\varphi(\bar{x}, a_\eta)\}_{\eta\in X_0} \cup \Gamma(\bar{x}, (a_\eta)_{\eta\in X_0})$

is consistent, then $\dim(\{\varphi(\bar{x}, a_\eta)\}_{\eta\in X_0}) = |x|$. Suppose not. Since φ witnesses SOP_3, we have $\bar{b} \models \{\varphi(\bar{x}, a_\eta)\}_{\eta\in X_0}$. Since Φ is not consistent, there exist $i < n$ and $\sigma(x_i; \check{x}^i, \bar{y}) \in \mathcal{L}$ such that

* $\bar{b} \models \sigma(x_i; \check{x}^i, \bar{a}_{\bar{q}}, \bar{a}_{\bar{q}'}) \wedge \exists^! x\sigma(x, \check{x}^i, \bar{a}_{\bar{q}}, \bar{a}_{\bar{q}'})$ (note that we assume acl = dcl),

for some $\bar{q}$ is a tuple of rational numbers smaller than 0. Similarly, $\bar{q}' > 0$. Let

* $\varphi'(\check{x}^i, y, \bar{y}, \bar{y}') := \exists x_i(\varphi(\bar{x}, y) \wedge \sigma(x_i; \check{x}^i, \bar{y}, \bar{y}') \wedge \exists^{<k} x\sigma(x, \check{x}^i, \bar{y}), \bar{y}')$
* $a^*_{i,0} := (a_{r_{2i},0}, \bar{a}_{\bar{q}}, \bar{a}_{\bar{q}'})$ and $a^*_{i,1} := (a_{r_{2i+1},1}, \bar{a}_{\bar{q}}, \bar{a}_{\bar{q}'})$ for each $i < \omega$,

where $(r_i)_{i<\omega}$ is an increasing sequence of rational numbers between $\bar{q}, \bar{q}'$. Then $\varphi'(\check{x}^i, y, \bar{y}, \bar{y}')$ witness SOP_3 with $(a^*_{i,j})_{i<\omega,j<2}$. We can repeat this process until we obtain a witness satisfying n-variable theorem.

Note that we can not find $X \subseteq \{(i,j)\}_{i<\omega,j<2}$ with $|X| = \omega$ such that $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for each $\eta \neq \nu \in X$.

Proof

Theorem [K.]

If acl satisfies exchange property, then the following are equivalent.

(i) T has CTP.

(ii) there exist $\varphi(x,y)$ and $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that
- $\{\varphi(x,a_\eta)\}_{\eta\in X}$ is consistent for any descending comb X,
- $\{\varphi(x,a_\eta)\}_{\eta\in X}$ is inconsistent for any path X,
- $\dim(\{\varphi(x,a_\eta)\}_{\eta\in X})=|x|$ for any descending comb X.

(iii) there exist $\varphi(x,y)$ and $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that
- $\dim(\{\varphi(x,a_\eta)\}_{\eta\in X})=|x|$ for any descending comb X.
- $\dim(\{\varphi(x,a_\eta)\}_{\eta\in X})<|x|$ for any path X.

The similar statements hold for BTP and SOP_3. For the SOP_3 case, we need acl = dcl.

(i)⇒(ii) and (ii)⇒(iii) are clear. To show (iii)⇒(i), consider

$$\Phi((x_\eta)_{\eta\in\omega^{\le n}},y):=\{\varphi(x_{\eta_0},\dots,x_{\eta_{n-1}},y):\emptyset\lhd\eta_0\lhd\cdots\lhd\eta_{n-1}\}\cup\{x_\emptyset=x_\emptyset\}.$$

Then $\cup_{\eta\in X}\Phi((x_\eta)_{\eta\in\omega^{\le n}},a_\eta)$ is consistent for each descending comb X, inconsistent for each path X, by using s-modeling property and s-indiscernibility on $(x_\eta)_{\eta\in\omega^{\le n}}$. By compactness, we can find a witness of CTP.

Proof

Proof of that T is NCTP if and only if T^{gt} is NCTP

Fix a saturated model $(\mathbb{M}, H(\mathbb{M})) \models T^{\mathrm{ind}}$. Then $\mathbb{M} \models_{\mathcal{L}} T$ and $H(\mathbb{M}) \models_{\mathcal{L}_R} T^{\mathrm{gt}}$. If T has CTP, then by n-variable theorem, there exist $\varphi(x, y)$ and $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

* φ witnesses CTP with $(a_\eta)_{\eta \in \omega^{<\omega}}$,
* $\dim(\{\varphi(x, a_\eta)\}_{\eta \in X}) = |x|$ for any descending comb X.

By density property, we may assume $(a_\eta)_{\eta \in \omega^{<\omega}} \subseteq H(\mathbb{M})$. It is known that in T^{ind}, for every set $D \subseteq \mathbb{M}^n$ definable over $\mathrm{acl}(H(\mathbb{M}))$, there is a subset $D' \subseteq D$ definable over $H(\mathbb{M})$ such that $\dim(D \setminus D') < n$. Using this, we can find $\varphi'(x, y)$ and $(a'_\eta)_{\eta \in \omega^{<\omega}} \subseteq H(\mathbb{M})$ such that

$$\dim(\varphi(x, a_\eta) \wedge \neg\varphi'(x, a'_\eta)) < |x|. \quad (*)$$

Thus $\dim_{H(\mathbb{M})}(\{\varphi'(x, a'_\eta)\}_{\eta \in X}) = |x|$ for each descending comb X by applying density property. By $(*)$, $\dim(\{\varphi'(x, a'_\eta)\}_{\eta \in X}) < |x|$ for any path X and hence $\dim_{H(\mathbb{M})}(\{\varphi'(x, a'_\eta)\}_{\eta \in X}) < |x|$. Thus T^{gt} has CTP by the second version of n-variable theorem.

Conversely, if T^{gt} has CTP, then there exist $\varphi(x, y)$ and $(a_\eta)_{\eta \in \omega^{<\omega}} \subseteq H(\mathbb{M})$ such that φ witnesses CTP with $(a_\eta)_{\eta \in \omega^{<\omega}}$ and $\dim(\{\varphi(x, a_\eta)\}_{\eta \in X}) = |x|$ for any descending comb X. $\dim(\{\varphi(x, a_\eta)\}_{\eta \in X}) < |x|$ in $\mathbb{M}$ for any path X by density property. Clearly $\dim(\{\varphi(x, a_\eta)\}_{\eta \in X}) = |x|$ in $\mathbb{M}$ for any descending comb X. By the second n-variable theorem, T has CTP.

Proof

For a unary predicate $H(x)$ and n-tuple of single variables $\bar{x} := (x_0, ..., x_{n-1})$, we put $H(\bar{x}) := \bigwedge_{i<n} H(x_i)$.
We define the *algebraic dimension over H* by $\dim^H(\bar{b}/A) := \dim(\bar{b}/AH(\mathbb{M}))$.
Note that $AH(\mathbb{M})$ does not have to be small.
The following statement is useful when we consider expansions by adding unary predicate symbol.

Theorem [K.]

Let $\mathcal{L}$ be a language, T an $\mathcal{L}$-theory, and H a unary predicate such that $H \in \mathcal{L}$. If T has CTP, then there exist $\varphi(\bar{x}, \bar{x}', y)$ and $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

- $\varphi(\bar{x}, \bar{x}', y) \wedge \neg H(\bar{x}) \wedge H(\bar{x}')$ witnesses CTP with $(a_\eta)_{\eta \in \omega^{<\omega}}$,
- $\dim(\{\varphi(\bar{x}, \bar{x}', a_\eta)\}_{\eta \in X}) = |\bar{x}| + |\bar{x}'|$ for any descending comb X.
- $\dim^H(\{\varphi(\bar{x}, \bar{x}', a_\eta)\}_{\eta \in X}) = |\bar{x}|$ for any descending comb X.

The similar statements hold for BTP and SOP_3. For the SOP_3 case, we need acl = dcl.

Let $\mathcal{L}$ be a language and $\mathcal{L}_H := \mathcal{L} \cup \{H\}$ an extended language. In an expanded $\mathcal{L}_H$-structure on an $\mathcal{L}$-structure $\mathbb{M}$, if $H(\mathbb{M})$ has density/codensity property and the expansion $(\mathbb{M}, H(\mathbb{M}))$ is not that 'complex', then we can apply n-variable theorem on the preservation problem. (continue)

Proofs

(continue)

It is known that in a lovely pair or a H-stucture expansion $(\mathbb{M}, H(\mathbb{M}))$ on an $\mathcal{L}$-structure $\mathbb{M}$, for H-independent tuples $\bar{a}$ and $\bar{b}$,

$\mathrm{tp}_{\mathcal{L}}(\bar{a}) = \mathrm{tp}_{\mathcal{L}}(\bar{b}) \wedge \mathrm{qftp}_{\{H\}}(\bar{a}) = \mathrm{qftp}_{\{H\}}(\bar{b})$ if and only if $\mathrm{tp}_{\mathcal{L}_H}(\bar{a}) = \mathrm{tp}_{\mathcal{L}_H}(\bar{b})$.

($\bar{a}$ is H-independent if $\dim_{\mathcal{L}}(\bar{a}/H(\bar{a}) = \dim^{H}_{\mathcal{L}}(\bar{a}/H(\bar{a}))$. It is easy to obtain so if a witness of CTP $(\varphi, (a_\eta)_{\eta \in \omega^{<\omega}}$ is given, then we may assume a_η is H-independent for all $\eta \in \omega^{<\omega}$.)

Thus if $\dim_{\mathcal{L}}(\{\varphi(x, a_\eta)\}_{\eta \in X}) = |x|$ and a quantifier free type $q(\bar{x})$ in the language $\{H\}$ is given, then we can find $\bar{b} \models \{\varphi(x, a_\eta)\}_{\eta \in X}$ such that

- $\dim_{\mathcal{L}_H}(\bar{b}/(a_\eta)_{\eta \in X}) = |x|$,
- $\bar{b} \models q(\bar{x})$.

This is a key idea of the following observation.

Theorem [K.]

Let T be a geometric theory and T^* be T_P or T^{ind}.

- T is NCTP if and only if T^* is NCTP.
- T is NBTP if and only if T^* is NBTP.
- If acl = dcl, then T is NSOP_3 if and only if T^* is NSOP_3.

Questions

Let $\mathbb{F}$ be a field with characteristic 0, $\mathcal{L} \supseteq \mathcal{L}_{\mathbb{F}-mod}$, and V be an $\mathcal{L}$-structure whose restriction on $\mathcal{L}_{\mathbb{F}-mod}$ is a vector space over $\mathbb{F}$. Fix a subring $R \subseteq \mathbb{F}$ and for each $\varphi(\bar{x}) \in \mathcal{L}_{R-mod}$, let $P_\varphi(\bar{x})$ be a new predicate symbol. For a new unary predicate G, let $\mathcal{L}_G := \mathcal{L} \cup \{P_\varphi : \varphi \in \mathcal{L}_{R-mod}\} \cup \{G\}$ and consider an $\mathcal{L}_G$-expansion (V, G) on V such that

- G is a proper R-submodule of V,
- G has dense/codense property,
- For $\mathrm{Frac}(R)$-linearly independent $\lambda_0, ..., \lambda_{n-1} \in \mathbb{F}$ and $g_0, ..., g_{n-1} \in G$, if $\lambda_0 g_0 + \cdots + \lambda_{n-1} g_{n-1} = 0$, then $\bar{g} = \bar{0}$.

Question

Suppose T_G has CTP (BTP, SOP_3). Does T also have CTP (BTP, SOP_3)?

It is known that for G-independent (similar to H-independent) tuples $\bar{a}, \bar{b}$,

$$\mathrm{tp}_{\mathcal{L}}(\bar{a}) = \mathrm{tp}_{\mathcal{L}}(\bar{b}) \wedge \mathrm{qftp}_{\{G\}}(\bar{a}) = \mathrm{qftp}_{\{G\}}(\bar{b}) \wedge \mathrm{tp}^G_{R-mod}(G(\bar{a})) = \mathrm{tp}^G_{R-mod}(G(\bar{b}))$$

if and only if

$$\mathrm{tp}_{\mathcal{L}_H}(\bar{a}) = \mathrm{tp}_{\mathcal{L}_H}(\bar{b}).$$

To show the preservation results on T_G, we need a stronger n-variable theorem.

Questions

Question

For a henselinan valued field $\mathcal{K}$ with characteristic $(0,0)$ in the Denef-Pas language, if $\mathrm{Th}(\mathcal{K})$ has CTP (BTP, SOP_3), then does $\mathrm{Th}(k)$ also have CTP (BTP, SOP_3)?

Question

For ACF_T, the theory of algebraically closed field with a distinguished subfield of T, if ACF_T has CTP (BTP, SOP_3), then does T also have CTP (BTP, SOP_3)?

Question

Does CTP (BTP, SOP_3) have one-variable theorem?

References

[1] Gabriel Day, Results on Colored Tree Properties

[2] Christian d'Elbée, Itay Kaplan, Leor Neuhauser, On algebraically closed fields with a distinguished subfield

[3] Artem Chernikov, Theories without the tree property of the second kind,

[4] Nicholas Ramsey, A note on NSOP_1 in one variable

[5] Alexander Berenstein and Evgueni Vassiliev, Generic trivializations of geometric theories

[6] Alexander Berenstein and Hyeung-Joon Kim, Dense codense predicates and the NTP_2

[7] Jan Dobrowolski and Hyeungjoon Kim, A preservation theorem for theories without the tree property of the first kind

[8] Artem Chernikov, Nicholas Ramsey, On the model-theoretic tree property

[9] James Hanson, Bi-invariant types, reliably invariant types, and the comb tree property

[10] Alex Kruckman, Nicholas Ramsey, A new Kim's Lemma

[11] Saharon Shelah, Classification theory and the number of non-isomorphic models

[12] JinHoo Ahn and Joonhee Kim, SOP_1, SOP_2, and antichain tree property

[13] JinHoo Ahn, Joonhee Kim, and Junguk Lee, On the Antichain Tree Property

[14] Scott Mutchnik, On NSOP_2 theories

[15] Alexander Berenstein, Christian d'Elbée, and Evgueni Vassiliev, Vector spaces with a dense-codense generic submodule,

[16] JinHoo Ahn, Joonhee Kim, Junguk Lee, and Hyoyoon Lee, Preservation of NATP