Forking, Dividing, Morley Sequences, and Independence Notions in Model Theory

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References

[1] Gabriel Day, Results on colored tree properties

[2] Christian d'Elbée, Axiomatic theory of independence relations in model theory

[3] Saharon Shelah, Classification theory and the number of non-isomorphic models

[4] Byunghan Kim and Anand Pillay, Simple theories

[5] Itay Kaplan and Nicholas Ramsey, On Kim-independence

[6] Hans Adler, Explanation of independence

[7] Victor Harnik and Leo Harrington, Fundamentals of forking

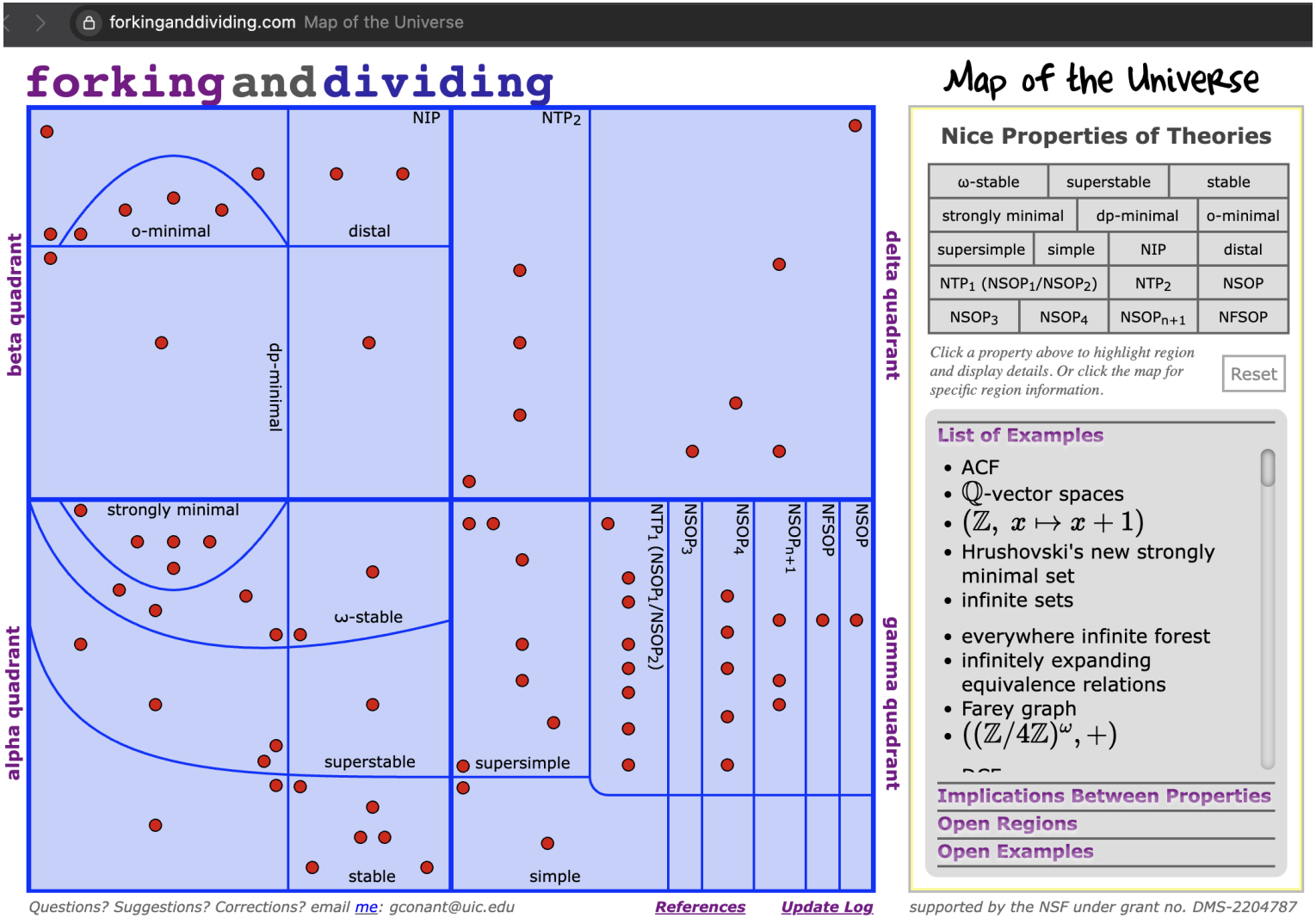
[8] Byunghan Kim, Simplicity theories

One of goals of model theory is understanding mathematical structures via first-order languages. If we fix a mathematical structure $\mathbb{M}$ and a language $\mathcal{L}$, then we obtain the theory $\mathrm{Th}_{\mathcal{L}}(\mathbb{M})$, the set of all $\mathcal{L}$-sentences true in $\mathbb{M}$.

If $\mathbb{M} \cong_{\mathcal{L}} \mathbb{N}$ (that is, there exists a bijection $\sigma : \mathbb{M} \to \mathbb{N}$ preserving all symbols in $\mathcal{L}$, e.g., $\sigma(R^{\mathbb{M}}) = R^{\mathbb{N}}$), then $\mathrm{Th}_{\mathcal{L}}(\mathbb{M}) = \mathrm{Th}_{\mathcal{L}}(\mathbb{N})$. Although the converse is not true in general, many features of a structure are still determined by its theory.

So we want to find reasonable ways of classifying first-order theories.

- Existence a family of definable sets satisfying some consistency-inconsistency conditions
- Existence of a certain pre-independence relation
- Using collapsing indiscernible
- Using a rank notion

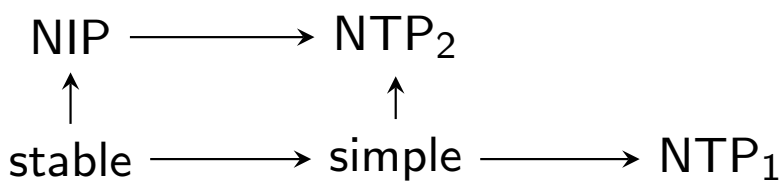


$$\begin{array}{ccccc} \text{NIP} & \longrightarrow & \text{NTP}_2 \\ \uparrow & & \uparrow \\ \text{stable} & \longrightarrow & \text{simple} & \longrightarrow & \text{NTP}_1 \end{array}$$

Simplicity, NTP_1, and NTP_2 can be defined by using the following consistency-inconsistency configurations.

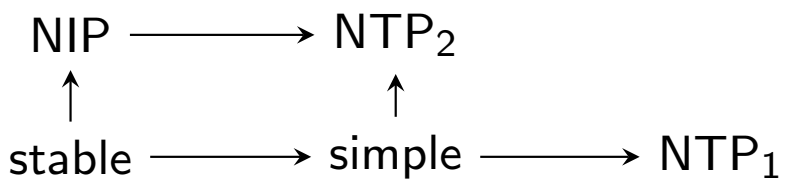
Definition

- We say a first-order theory T ($\mathbb{M} \models T$) is *simple* if there do not exist $\varphi(x,y)$ and $\{a_\eta \in \mathbb{M}^{|y|} : \eta \in \omega^{<\omega}\}$ with $|a_\eta| = |y|$ for all $\eta \in \omega^{<\omega}$ such that
  - $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent for all path $X \subseteq \omega^{<\omega}$,
  - $\{\varphi(x, a^i_\eta) : i < \omega\}$ is inconsistent for all $\eta \in \omega^{<\omega}$.
- T is *NTP_1* if there do not exist $\varphi(x,y)$ and $\{a_\eta \in \mathbb{M}^{|y|} : \eta \in \omega^{<\omega}\}$ with $|a_\eta| = |y|$ for all $\eta \in \omega^{<\omega}$ such that
  - $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent for all path X,
  - $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all incomparable η, ν.
- T is *NTP_2* if there do not exist $\varphi(x,y)$ and $\{a_{i,j} \in \mathbb{M}^{|y|} : i,j < \omega\}$ with $|a_{i,j}| = |y|$ for all $\eta \in \omega^{<\omega}$ such that
  - $\{\varphi(x, a_{i,f(i)}) : i < \omega\}$ is consistent for all map f from ω to ω,
  - $\{\varphi(x, a_{i,j}), \varphi(x, a_{i,k'})\}$ is inconsistent for all $i < \omega$ and $j, j' < \omega$.



Definition

- We say T is *stable* if there do not exist $\varphi(x, y)$, $(a_i)_{i \in \omega}$, and $(b_i)_{i \in \omega}$ such that $\models \varphi(a_i, b_j) \Leftrightarrow i < j$.
- We say T is *NIP* if there do not exist $\varphi(x, y)$, $(a_i)_{i \in \omega}$, and $(b_I)_{I \subseteq \omega}$ such that $\models \varphi(a_i, b_j) \Leftrightarrow i \in I$.



n-colored linear order

For each $n < \omega$, let $\mathcal{L}_{n,<} := \{C_0, ..., C_{n-1}, <\}$ where C_i is unary predicate for each $i < n$. Let $\mathcal{K}_{n,<}$ be a class of finite $\mathcal{L}_{n,<}$-structures A such that $C_0(A) \dot{\cup} \cdots \dot{\cup} C_{n-1}(A) = A$ and $<$ is a linear order on A. Then $\mathcal{K}_{n,<}$ is a Fraïssé class. Let $\mathbf{c}_n$ is the Fraïssé limit of $\mathcal{K}_{n,<}$.

Definition [Day]

T is $\mathbf{c}_n$-*NTP* if there do not exist $\varphi(x, y) \in \mathcal{L}$, $n < \omega$, $(a_\eta)_{\eta \in \mathbf{c}_n^{<\omega}}$, and $c_0, ..., c_{k-1} \in \mathbf{c}_n$ such that

- $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for any path X,
- $\{\varphi(x, a_{\eta^\frown(c_0')}), ..., \varphi(x, a_{\eta^\frown(c_{k-1}')})\}$ is inconsistent for $c_0'...c_{k-1}' \equiv c_0...c_{k-1}$,

T is $\mathbf{c}_n$-*NTP*$_2$ if and only if there do not exist $\varphi(x, y) \in \mathcal{L}$, $n < \omega$, $(a_{i,c})_{i<\omega, c \in \mathbf{c}_n}$, and $c_0, ..., c_{k-1} \in \mathbf{c}_n$ such that

- $\{\varphi(x, a_{i,f(i)})\}_{i<\omega}$ is consistent for any function $f : \omega \to \mathbf{c}_n$,
- $\{\varphi(x, a_{i,c_0'}), ..., \varphi(x, a_{i,c_{k-1}'})\}$ is inconsistent for any $i < \omega$ and $c_0'...c_{k-1}' \equiv c_0...c_{k-1}$,

Fact [Day]

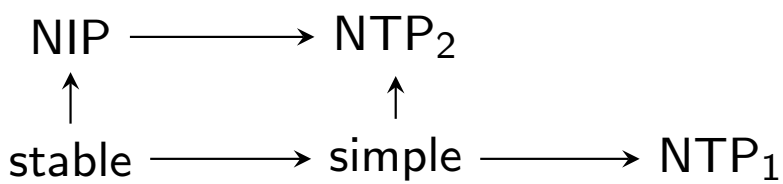
T is stable if and only if it is $\mathbf{c}_2$-NTP. T is NIP if and only if it is $\mathbf{c}_2$-NTP$_2$.

In fact, stability has many different characterizations.

Fact

The following are equivalent.

- T is stable.
- T is $\mathbf{c}_2$-NTP.
- There is κ such that $|S_1(A)| \leq \kappa$ for all $A \subseteq \mathbb{M}$ with $|A| = \kappa$.
- Every indiscernible sequence is totally indiscernible.
- There exists a pre-independence relation satisfying normality, monotonicity, base monotonicity, transitivity, right extension, symmetry, strong finite character, and stationary.



We call a ternary relation $\square \downarrow_{\square} \square$ a *pre-independence relation* if $A \downarrow_C B \Leftrightarrow \sigma(A) \downarrow_{\sigma(C)} \sigma(B)$ for all A, B, C and an automorphism σ.

Definition

Let $\downarrow$ be a pre-independence relation.

- (normality) $A \downarrow_C B \Rightarrow A \downarrow_C BC$
- (monotonicity) $A \downarrow_C B, A' \subseteq A, B' \subseteq B \Rightarrow A' \downarrow_C B'$
- (base monotonicity) $A \downarrow_D B, D \subseteq C \subseteq B \Rightarrow A \downarrow_C B$
- (transitivity) $A \downarrow_D C, A \downarrow_C B,\ D \subseteq C \subseteq B \Rightarrow A \downarrow_D B$
- (right extension) $A \downarrow_C B, B \supseteq B' \Rightarrow \exists B'' \equiv_{CB} B'$ such that $A \downarrow_C B'$
- (symmetry) $A \downarrow_C B \Rightarrow B \downarrow_C A$
- (strong finite character) If $A \not\downarrow_C B$, then there exists $\varphi(x, b) \in \mathrm{tp}(A/BC)$ such that $a' \models \varphi(x, b) \Rightarrow a' \not\downarrow_C B$
- (finite character) If $A \not\downarrow_C B$, then there exist $A_0 \subseteq A, B_0 \subseteq B$ with $|A_0|, |B_0| < \omega$ such that $A_0 \not\downarrow_C B_0$
- (stationary) For a model $M \prec \mathbb{M}$ and $A, A', B \subseteq \mathbb{M}$, if $A \downarrow_M B, A' \downarrow_M B, A \equiv_M A'$, then $A \equiv_{MB} A'$
- (existence) $A \downarrow_C C$ for all A, C
- (full existence) For all A, B, C, $\exists A' \equiv_C A\ (A' \downarrow_C B)$
- (local character) For all A, B, $\exists B_0 \subseteq B (|B_0| < |T| \wedge A \downarrow_{B_0} B)$

Definition

$\mathrm{acl}(A)$ is the set of all b such that there exists $\varphi(x, a) \in \mathrm{tp}(b/A)$ such that $\models \exists^{<n} x \varphi(x, a)$.
$\varphi(x, a)$ is said to be *algebraic*, b is said to be *algebraic over A* and $\mathrm{acl}(A)$ is called the *algebraic closure of A*. We define pre-independence relations $\mathop{\smile\!\!\!\!\!\!|}^{a}$, $\mathop{\smile\!\!\!\!\!\!|}^{u}$, and $\mathop{\smile\!\!\!\!\!\!|}^{h}$ as follows.

- $A \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} B$ if $\mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C)$.
- $A \mathop{\smile\!\!\!\!\!\!|}^{u}_{C} B$ if $\mathrm{tp}(A/BC)$ is finitely satisfiable in C.
- $A \mathop{\smile\!\!\!\!\!\!|}^{h}_{C} B$ if $\varphi(x, b) \in \mathrm{tp}(A/BC) \Rightarrow \exists c \in C\, (\varphi(x, c) \in \mathrm{tp}(A/C))$

Note that $a \notin \mathrm{acl}(B)$ if and only if there are infinitely many automorphisms σ over $\mathrm{acl}(B)$ such that $\sigma(a) \neq a$.

Remark

Let X be a set and suppose G is a group of permutations of X. Let $P, Q \subseteq X$ be finite subsets such that no point in P has a finite orbit. Then there is some $g \in G$ such that $gP \cap Q = \emptyset$.

Fact

$\downarrow^a$ always satisfies symmetry, normality, monotonicity, finite character, transitivity, and full existence.

Symmetry, normality, monotonicity, finite character, and transitivity are clear.
(full existence) Fix A, B, C. To find $A' \equiv_C A$ such that $A' \downarrow^a_C B$, it is enough to show that $P(\bar{x}) \cup \{x \neq b : x \in \bar{x}, b \in B^*\}$ is consistent, where $P(\bar{x}) := \mathrm{tp}(A^*/\mathrm{acl}(C))$, $A^* := \mathrm{acl}(AC) \setminus \mathrm{acl}(C)$, $B^* := \mathrm{acl}(BC) \setminus \mathrm{acl}(C)$.
By compactness, it is enough to show that $P_0(\bar{x}_0) \cup \{x \neq b : x \in \bar{x}_0, b \in B_0^*\}$ is consistent, for some $P_0(\bar{x}_0) := \mathrm{tp}(A_0^*/\mathrm{acl}(C))$, $A_0^* \subseteq A^*, B_0^* \subseteq B^*, |A_0^*|, |B_0^*| < \omega$.
We can apply the Remark on $P = A_0^*, Q = B_0^*, G = Aut(\mathbb{M}/\mathrm{acl}(C))$.

Definition

- We say $\varphi(x, b)$ *divides* over C if there is an indiscernible sequence $(b_i)_{i<\omega}$ over C with $b_0 = b$ such that $\{\varphi(x, b_i)\}_{i<\omega}$ is inconsistent.
- We say $\varphi(x, b)$ *forks* over C if there exist dividing formulas $\psi_0(x, b_0), ..., \psi_{n-1}(x, b_{n-1})$ over C such that $\varphi(x, b) \vdash \bigvee \psi_i(x, b_i)$.

Using this we can define $\downarrow^d$ and $\downarrow^f$.

- $A \downarrow^d_C B$ if $\mathrm{tp}(A/BC)$ has no dividing formula over C.
- $A \downarrow^f_C B$ if $\mathrm{tp}(A/BC)$ has no forking formula over C.

Fact

- $\downarrow^d$ always satisfies existence, strong finite character, monotonicity, normality, base monotonicity, and transitivity.
- $\downarrow^f$ always satisfies strong finite character, monotonicity, normality, base monotonicity, transitivity, and right extension.
- $a \downarrow^u_C b \Rightarrow a \downarrow^f_C b \Rightarrow a \downarrow^d_C b \Rightarrow a \downarrow^a_C b$ for all a, b, C.

Fact

- If there exists a pre-independence relation $\downarrow$ satisfying normality, monotonicity, base monotonicity, local character, transitivity, right extension, symmetry, strong finite character, and stationary, then T is stable and $\downarrow = \downarrow^f$.
- If T is stable, then $\downarrow^f$ satisfies symmetry and stationary.

Definition

Let $\downarrow$ be a pre-independence relation.

- (independence theorem) For a model $M \prec \mathbb{M}$ and $A, A', B, C \subseteq \mathbb{M}$, if $B \downarrow_M C, A \downarrow_M B, A' \downarrow_M C, A \equiv_M A'$, then $\exists A''$ such that $A'' \downarrow_M BC$, $A'' \equiv_{MB} A$, $A'' \equiv_{MC} A'$

Fact 2

(i) If there exists a pre-independence relation $\downarrow$ satisfying normality, monotonicity, base monotonicity, local character, transitivity, right extension, symmetry, strong finite character, and independence theorem, then T is simple and $\downarrow = \downarrow^f$.

(ii) If T is simple, then $\downarrow^f$ satisfies symmetry and independence theorem.

Definition

Let $\downarrow$ be a pre-independence relation.

- We say a sequence $(b_i)_{i<\omega}$ is a $\downarrow$-*Morley sequence* over C if it is indiscernible over C and $b_i \downarrow_C b_{<i}$ for all $i < \omega$, where $b_{<i} := \{b_j : j < i\}$.
- We say $\varphi(x, b)$ $\downarrow$-*Kim-divides* over C if there is a $\downarrow$-Morley sequence $(b_i)_{i<\omega}$ over C with $b_0 = b$ such that $\{\varphi(x, b_i)\}_{i<\omega}$ is inconsistent.
- We say $\varphi(x, b)$ $\downarrow$-*Kim-forks* over C if there exist $\downarrow$-Kim-dividing formulas $\psi_0(x, b_0), ..., \psi_{n-1}(x, b_{n-1})$ over C such that $\varphi(x, b) \vdash \bigvee \psi_i(x, b_i)$.

As in the case of $\downarrow^d$ and $\downarrow^f$, we can define $\downarrow^d$ and $\downarrow^f$.

- $A \downarrow_C^{Kd\downarrow} B$ if $\mathrm{tp}(A/BC)$ has no $\downarrow$-Kim-dividing formula over C.
- $A \downarrow_C^{K\downarrow} B$ if $\mathrm{tp}(A/BC)$ has no $\downarrow$-Kim-forking formula over C.

Notation

$\downarrow^{Kd} := \downarrow^{Kd^{\downarrow f}}$, $\downarrow^{K} := \downarrow^{K^{\downarrow f}}$

Definition

Let $\downarrow$ be a pre-independence relation satisfying strong finite character. Let M be a model. We say $\downarrow$ satisfies witnessing property if it satisfies the following.

- (witnessing) If $a \not\downarrow_M b$, $\varphi(x, mb) \in \mathrm{tp}(a/Mb)$, and $a' \not\downarrow_M b$ for all $a' \models \varphi(x, mb)$, then $\{\varphi(x, b_i)\}_{i<\omega}$ is inconsistent for any $\downarrow^f$-Morley sequence $(b_i)_{i<\omega}$ over M with $b_0 = b$.

Fact

- If there exists a pre-independence relation $\downarrow$ satisfying strong finite character (over models), existence (over models), monotonicity (over models), symmetry (over models), witnessing (over models), and independence theorem, then T is NTP_1 and $\downarrow = \downarrow^K$ over models.
- If T is NTP_1, then $\downarrow^K$ satisfies symmetry over models and independence theorem.

Fact 3

T is simple if and only if $\mathop{\downarrow}^d$ satisfies local character.

Proof of Fact 2 (i). First we show $\mathop{\downarrow}=\mathop{\downarrow}^d$. Suppose $a \not\mathop{\downarrow}_C b$.

[Step a1] $\mathop{\downarrow}$ satisfies existence. To show this fix d, E. We want to show $d \mathop{\downarrow}_E E$. By local character, there is $E_0 \subseteq E$ with $|E_0| < |T|$ such that $d \mathop{\downarrow}_{E_0} E$. By base monotonicity, we have $d \mathop{\downarrow}_E E$.

[Step a2] There exists a $\mathop{\downarrow}$-Morley sequence $(b_i)_{i<\kappa}$ over C with $b_0 = b$ where κ is sufficiently large and $\mathrm{cf}(\kappa) = \kappa$. We can show this by existence and right extension of $\mathop{\downarrow}$.

[Step a3] Let $p(x, y) := \mathrm{tp}(a, b/C)$. Then $\bigcup_{i<\kappa} p(x, b_i)$ is inconsistent. If not, then it is realized by a'. Thus $a' \not\mathop{\downarrow}_C b_i$ for all $i < \kappa$. By transitivity, symmetry, and monotonicity, $a' \not\mathop{\downarrow}_{Cb_{<i}} b_i$ for each $i < \kappa$. It yield a contradiction with local character. Thus $\bigcup_{i<\kappa} p(x, b_i)$ is inconsistent and $\mathrm{tp}(a/Cb)$ has a dividing formula over C. $a \not\mathop{\downarrow}^d_C b$.

(continue)

(continue)

Now we suppose $a \mathop{\downarrow}_C b$ and put $p(x,y) := \mathrm{tp}(a,b/C)$. To show $a \mathop{\downarrow^d_C} b$, it is enough to show that $\bigcup_{i<\kappa} p(x,b_i)$ is consistent for any indiscernible $(b_i)_{i<\omega}$ over C with $b_0 = b$. Fix any indiscernible $(b_i)_{i<\omega}$ over C with $b_0 = b$. By compactness we can extend it to $(b_i)_{i\leq\kappa}$ for some sufficiently large κ with $\mathrm{cf}(\kappa) = \kappa$.

[Step b1] We can find a sequence of small models $(M_i)_{i<\kappa}$ such that $Ab_{<i}M_{<i} \subseteq M_i$ for each $i < \kappa$, and $(b_j)_{i\leq j\leq\kappa}$ is indiscernible over M_i.

[Step b2] Take $M^* := \bigcup_{i<\kappa} M_i$. By local character, there is $i_0 < \kappa$ such that $b_\kappa \mathop{\downarrow}_{M_{i_0}} M^*$. By monotonicity, $b_\kappa \mathop{\downarrow}_{M_{i_0}} \{b_i : i_0 < i < \kappa\}$. Thus $(b_i)_{i_0<i<\kappa}$ is a $\mathop{\downarrow}$-Morley sequence over M_i.

[Step b3] By applying independence theorem repeatedly, we can find $a' \models \bigcup_{i_0<j<\kappa} p(x,b_j)$. This proves $a \mathop{\downarrow^d_C} b$ and hence T is simple by Fact 3.

In fact, if $\mathop{\downarrow^d}$ satisfies right extension, then $\mathop{\downarrow^d}=\mathop{\downarrow^f}$. Thus $\mathop{\downarrow}=\mathop{\downarrow^f}$.

Remark [Application of Kim-Pillay style criterion]

You can construct a structure $\mathbb{M}$ and define a pre-independence relation.
No matter how artificial your pre-independence relation is, once it satisfies Kim-Pillay style criterion for stability (or simplicity, NSOP_1), then $\mathrm{Th}(\mathbb{M})$ is stable (simple, NSOP_1, resp.)
Many examples have been shown to be NSOP_1 using this method. You can find several of them at `https://forkinganddividing.com/`, and some require no preliminaries.