n-variable theorems for positive consistency-inconsistency configurations and their applications to preservation problems

Joonhee Kim

Korea Institute for Advanced Study

BLAST 2026
May 20, Baylor University

First-order theories and dividing lines

By a *language*, we mean a set consisting of constant symbols, relation symbols, and function symbols.
Let $\mathcal{L}$ be a language. An *$\mathcal{L}$-term* is a sequence of elements of $\mathcal{L}$ and variables $x, y, z, \ldots$ generated according to the following rules.

- Every constant symbol $c \in \mathcal{L}$ is an $\mathcal{L}$-term.
- Every variable is an $\mathcal{L}$-term.
- If $t_0, ..., t_{n-1}$ are $\mathcal{L}$-terms and f is a function symbol in $\mathcal{L}$, then $f(t_0, ..., t_{n-1})$ is an $\mathcal{L}$-term.

An *$\mathcal{L}$-formula* is a sequence of elements of $\mathcal{L}$ and variables generated according to the following rules.

- If $t_0, ..., t_{n-1}$ are $\mathcal{L}$-terms and R is a relation symbol in $\mathcal{L}$, then $R(t_0, ..., t_{n-1})$ is an $\mathcal{L}$-formula
- If φ_0, φ_1 are $\mathcal{L}$-formulas and x is a variable appearing in φ_0, then $\neg\varphi_0$, $\varphi_0 \wedge \varphi_1$, $\varphi_0 \vee \varphi_1$, $\varphi_0 \rightarrow \varphi_1$, $\exists x \varphi_0$, and $\forall x \varphi_0$ are $\mathcal{L}$-formulas.

Let φ be an $\mathcal{L}$-formula. We say that a variable x is *free* in φ if it occurs in φ without being bound by $\forall$ or $\exists$. An *$\mathcal{L}$-sentence* is an $\mathcal{L}$-formula without free variables. An *$\mathcal{L}$-theory* is a set of $\mathcal{L}$-sentences that does not yield a contradiction. We say an $\mathcal{L}$-theory T is *complete* if $\sigma \in T$ or $\neg\sigma \in T$ for each $\mathcal{L}$-sentence σ.

First-order theories and dividing lines

Let $\mathcal{L} := \{(c_i)_{i\in I}, (R_j)_{j\in J}, (f_k)_{k\in K}\}$ be a language such that

- c_i is a constant symbol for each $i \in I$,
- R_j is a relation symbol with arity d_j for each $j \in J$,
- f_k is a function symbol with arity e_k for each $k \in K$.

By an *$\mathcal{L}$-structure*, we mean a tuple $\mathbb{M} := (M, (c_i^{\mathbb{M}})_{i\in I}, (R_j^{\mathbb{M}})_{j\in J}, (f_k^{\mathbb{M}})_{k\in K})$ such that

- M is a set,
- $c_i^{\mathbb{M}} \in M$ for each $i \in I$,
- $R_j^{\mathbb{M}} \subseteq M^{d_j}$ for each $j \in J$,
- $f_k^{\mathbb{M}} : M^{e_k} \to M$ for each $k \in K$.

For a given $\mathcal{L}$-structure $\mathbb{M}$, we can consider the $\mathcal{L}$-theory
$\mathrm{Th}_{\mathcal{L}}(\mathbb{M}) := \{\sigma \in \mathcal{L}\text{-sentence} : \sigma \text{ is true in } \mathbb{M} (\mathbb{M} \models \sigma\}$.
Conversely, if an $\mathcal{L}$-theory is given, then there always exists an $\mathcal{L}$-structure satisfying all sentences in the theory.

First-order theories and dividing lines

First-order theories can be classified in various ways. In other words, we can define classes of first-order theories in various ways. We will call these classes *dividing lines*.

Let T be a complete theory and M its model.

- [Shelah] We say T is *stable* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_i)_{i<\omega}$, and $(b_i)_{i<\omega}$ such that $\models \varphi(a_i, b_j)$ if and only if $i < j$.
- [Shelah] We say T has *NIP* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_i)_{i<\omega}$, and $(b_I)_{I\subseteq\omega}$ such that $\models \varphi(a_i, b_I)$ if and only if $i \in I$.
- [Shelah] We say T is *simple* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that

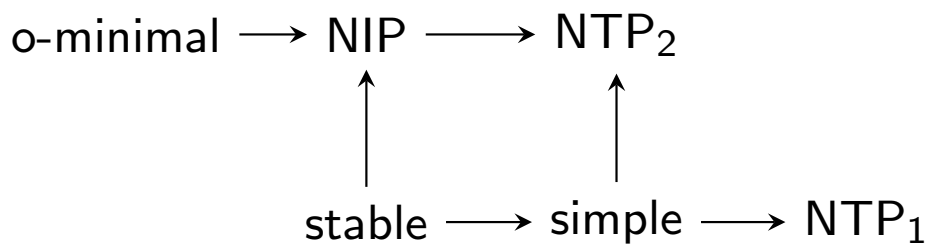
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta\in X}$ is consistent if X is a path in $\omega^{<\omega}$.
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta\in Y}$ is consistent if Y is a set of direct siblings.
- [Knight, Pillay, Steinhorn, van den Dries] We say T is *o-minimal* if it has a definable order $<$ and every definable set in M is a union on finitely many $<$-intervals and finitely many points.
- [Shelah] We say T is *NTP*$_1$ if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that

  $\{\varphi(\bar{x}, a_\eta)\}_{\eta\in X}$ is consistent if and only if X is a path in $\omega^{<\omega}$.
- [Shelah] We say T is *NTP*$_2$ if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_{i,j})_{i,j<\omega}$ such that

  $\{\varphi(\bar{x}, a_{i,f(i)})\}_{j<\omega}$ is consistent for all $f : \omega \to \omega$.
  $\{\varphi(\bar{x}, a_{i,j})\}_{j<\omega}$ is inconsistent for all $i < \omega$.

First-order theories and dividing lines

The following diagram is well known.



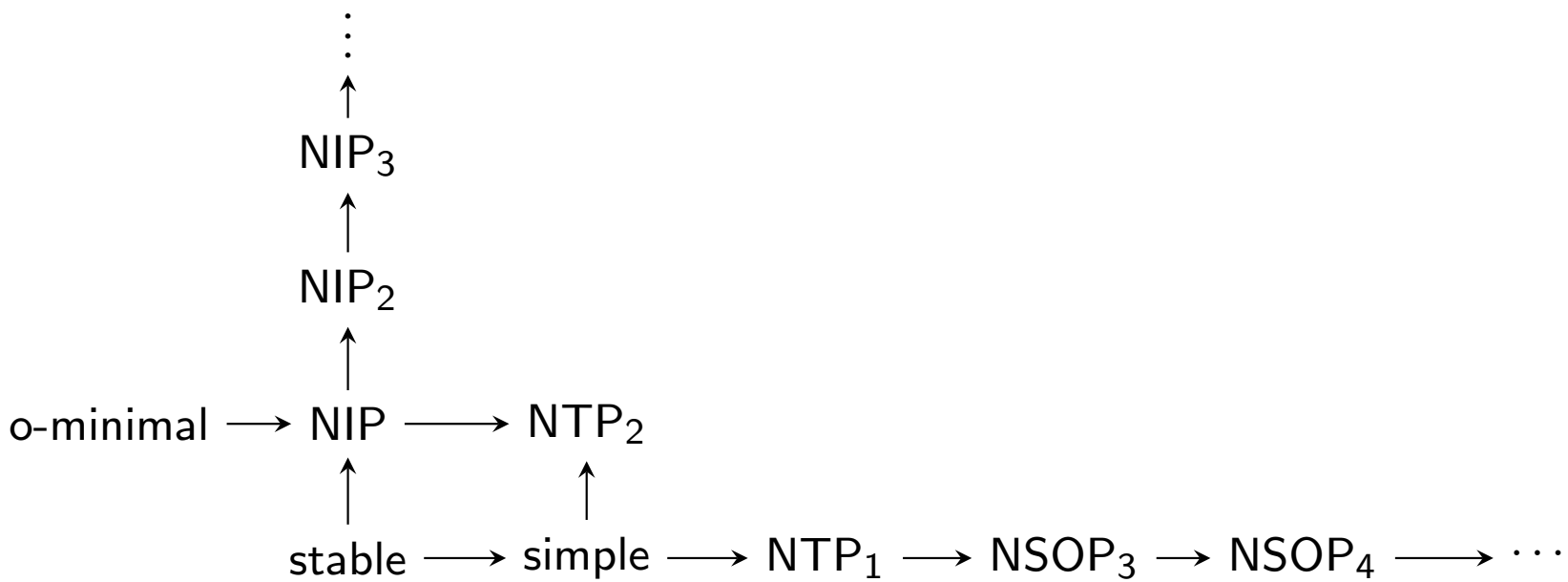
This diagram means that:

- if a theory is stable, then it is NIP and simple,
- if a theory is simple, then it is NTP_1 and NTP_2,
- if a theory is o-minimal, then it is NIP,
- if a theory is NIP, then it is NTP_2.

First-order theories and dividing lines

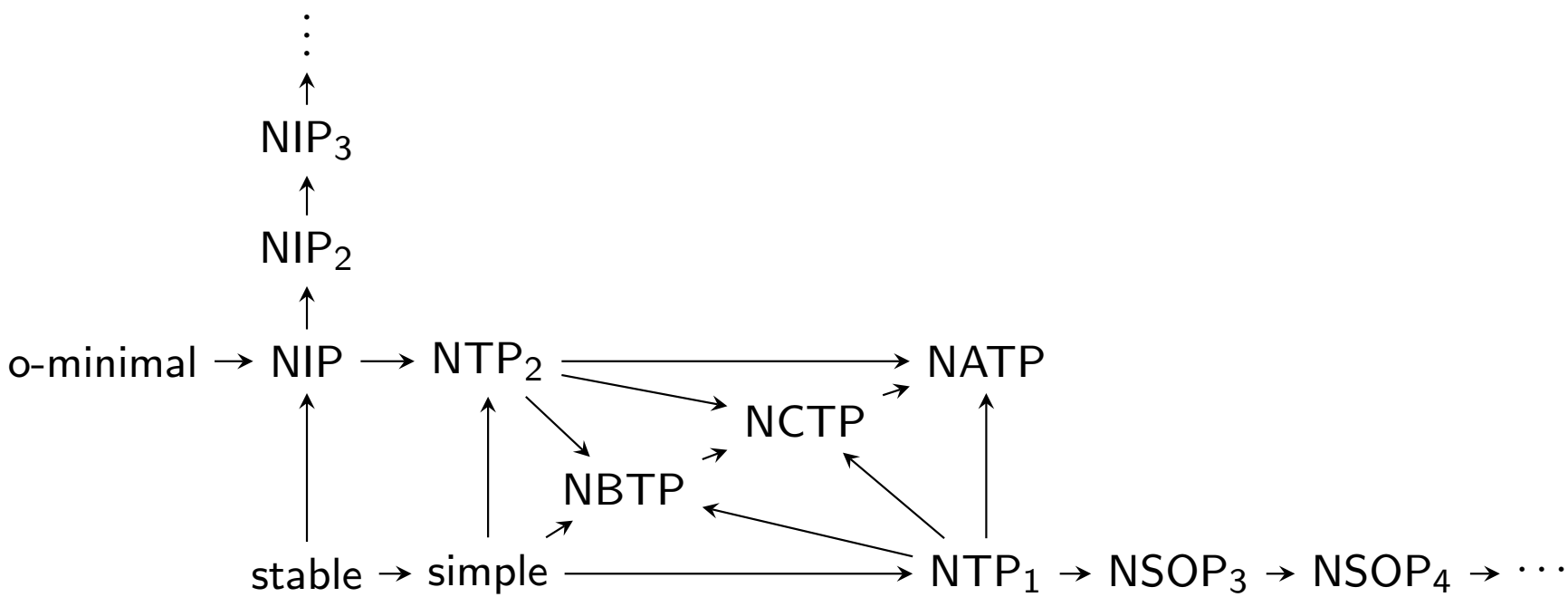
Let $2 < n < \omega$ and $1 < k < \omega$.

- [Shelah] We say T is $NSOP_n$ if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, a_{i+1})$ for all $i < \omega$ and $\{\varphi(\bar{x}_0, \bar{x}_1), ..., \varphi(\bar{x}_{n-2}, \bar{x}_{n-1}), \varphi(\bar{x}_{n-1}, \bar{x}_0)\}$ is inconsistent.
- [Shelah] We say T has NIP_k if there do not exist $\varphi(\bar{x}, \bar{y}_0, ..., \bar{y}_{k-1})$, $(a_i^0, ..., a_i^{k-1})_{i<\omega}$, and $(b_I)_{I \subseteq \omega^k}$ such that $\models \varphi(b_I, a_{i_0}^0, ..., a_{i_{k-1}}^{k-1})$ if and only if $(i_0, ..., i_{k-1}) \in I$.



First-order theories and dividing lines

- [Ahn, K.] We say T is *NATP* if there are no $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is inconsistent if and only if X is a path in $\omega^{<\omega}$.
- [Mutchnik] We say T is *NCTP* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent if X is a descending comb.
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in Y}$ is inconsistent if Y is a path.
- [Ramsey, Kruckman] We say T is *NBTP* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent if X is a left-leaning path.
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in Y}$ is inconsistent if Y is a right-veering path.



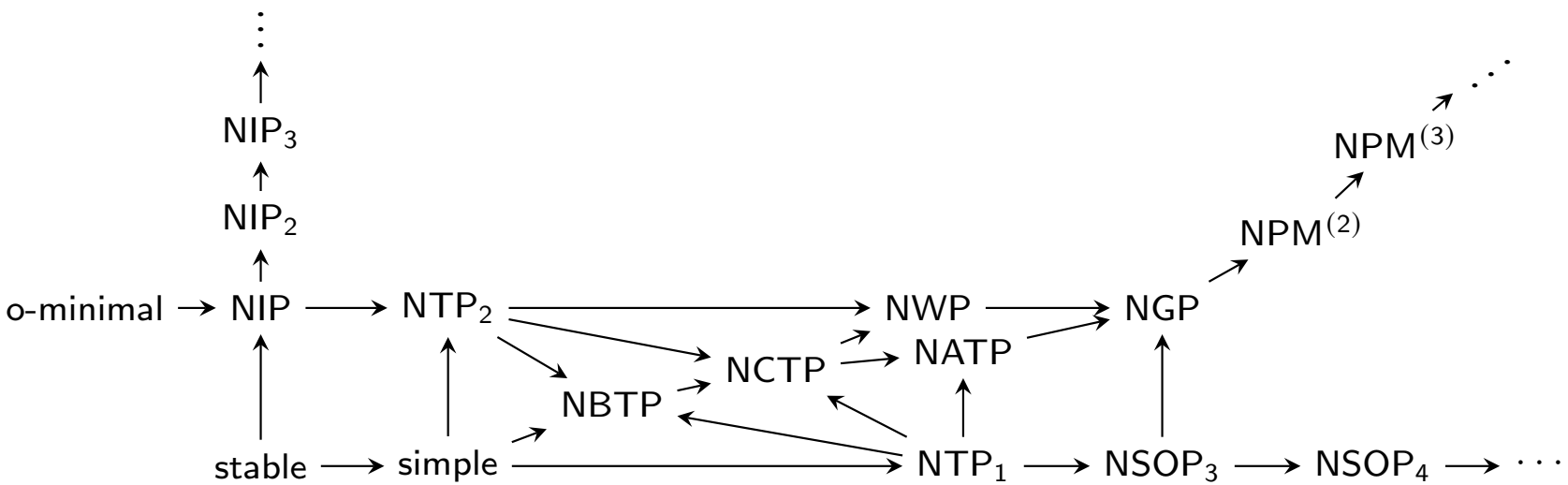
First-order theories and dividing lines

- [Hanson] For $k < \omega$, we say T is *NWP (having no $(k,1,1)$-weave of depth ω)* if there do not exist $\varphi(\bar{x}, \bar{y})$ and $(a_\eta)_{\eta \in (2^2)^\omega}$ such that

  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent for each finite right-1-comb X,
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is k-inconsistent for each finite up-1-comb X.

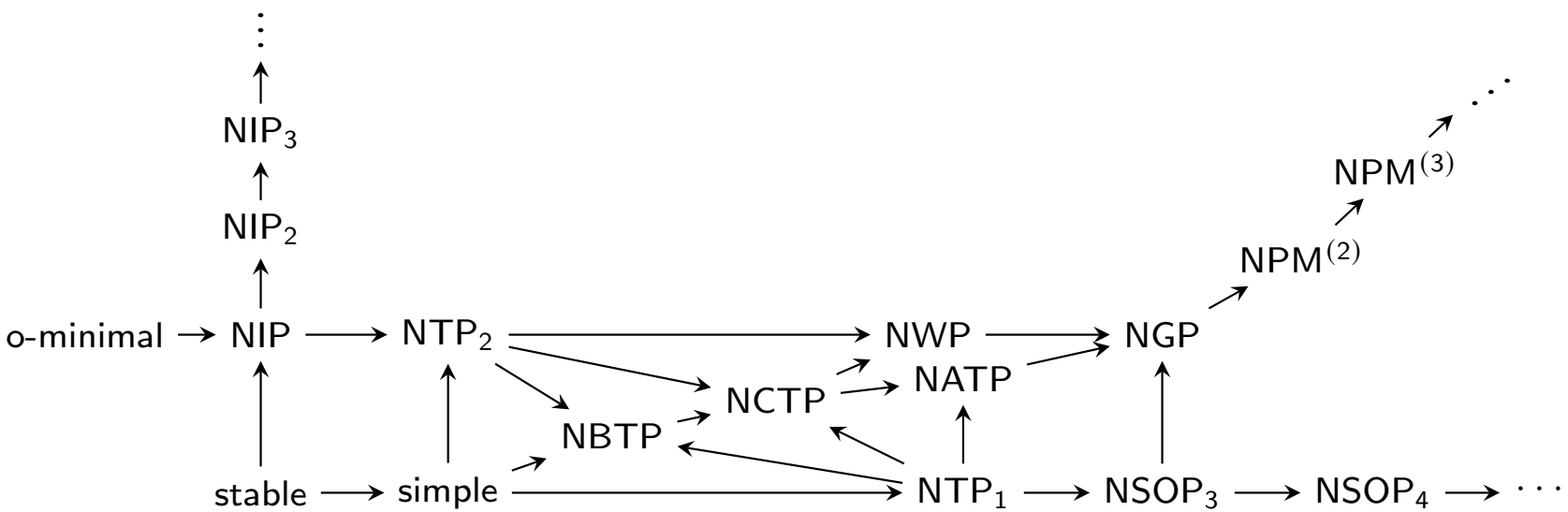
- [Hanson] For $k < \omega$, we say T has an *NGP (having no infinite k-grid)* if there do not exist $\varphi(\bar{x}, \bar{y})$ and $(a_{i,j})_{i,j<\omega}$ such that

  * $\{\varphi(\bar{x}, a_{i,j})\}_{(i,j) \in X}$ is consistent if X satisfies $(i,j), (i',j') \in X \Rightarrow i \leq i' \wedge j \leq j'$ or $i' \leq i \wedge j' \leq j$,
  * $\{\varphi(\bar{x}, a_{i,j})\}_{(i,j) \in X}$ is k-inconsistent if X satisfies $(i,j), (i',j') \in X \Rightarrow i \not\leq i' \vee j \not\leq j'$ and $i' \not\leq i \vee j' \not\leq j$.



First-order theories and dividing lines

- [Bailetti] For $1 < k < \omega$, we say T is *$NPM^{(k)}$ (having no $PM^{(k)}$)* if there do not exist $\varphi(\bar{x}, \bar{y})$ such that for a generic k-hypergraph (ω, E) on ω, there exists $(b_i)_{i<\omega}$ such that
  - $\{\varphi(\bar{x}, b_i)\}_{i \in X}$ is consistent if $X \subseteq \omega$ is a clique (i.e., $E(i_0, ..., i_{k-1})$ for all $i_0, ..., i_{k-1} \in X$),
  - $\{\varphi(\bar{x}, b_i)\}_{i \in X}$ is inconsistent if $X \subseteq \omega$ contains $i_0, ..., i_{k-1}$ such that $\neg E(i_0, ..., i_{k-1})$.



Generalized indiscernible sequences

Definition

Let $\mathcal{L}^*$ be a language and $\mathcal{I}^*$ an $\mathcal{L}^*$-structure. Let $\mathcal{L}$ be another language, T an $\mathcal{L}$-theory, $\mathbb{M}$ a monster model of T, and A a small subset of $\mathbb{M}$.

A family $(a_\eta)_{\eta \in \mathcal{I}^*}$ in $\mathbb{M}$ is said to be *$\mathcal{I}^*$-indiscernible over A* if for all finite tuples $\bar{\eta}, \bar{\nu}$ from $\mathcal{I}^*$,

$$\mathrm{qftp}_{\mathcal{L}^*}(\bar{\eta}) = \mathrm{qftp}_{\mathcal{L}^*}(\bar{\nu}) \quad \Rightarrow \quad \mathrm{tp}_{\mathcal{L}}(a_{\bar{\eta}}) = \mathrm{tp}_{\mathcal{L}}(a_{\bar{\nu}}).$$

Ordinary indiscernible sequences correspond to the case where $\mathcal{L}^* = \{<\}$ and $\mathcal{I}^*$ is a linear order.

Definition

Let $\mathcal{L}$ be an arbitrary language, T an arbitrary complete $\mathcal{L}$-theory, and $M \models T$ a sufficiently saturated model. Let $\mathcal{L}^*$ be a language and $\mathcal{I}^*$ an $\mathcal{L}^*$-structure. We say an $\mathcal{L}^*$-structure $\mathcal{I}^*$ has *modeling property* if for any $(a_i)_{i \in \mathcal{I}^*} \subseteq M$, there exists $\mathcal{I}^*$-indiscernible $(b_i)_{i \in \mathcal{I}^*}$ such that for any $\bar{i} \in \mathcal{I}^*$, there exists $\bar{j}$ such that $\mathrm{qftp}_{\mathcal{L}^*}(\bar{i}) = \mathrm{qftp}_{\mathcal{L}^*}(\bar{j})$ and $\mathrm{tp}_{\mathcal{L}}(b_{\bar{i}}) = \mathrm{tp}_{\mathcal{L}}(a_{\bar{j}})$.

The class of dividing lines satisfying n-variable theorem

Let $age_{\mathcal{L}^*}(\mathcal{I}^*)$ be the class of finitely generated $\mathcal{L}^*$-substructures of $\mathcal{I}^*$.

Definition

Let $\mathcal{L}^*$ be a language, $\mathcal{I}^*$ an $\mathcal{L}^*$-structure, κ a cardinal.

(i) We say $\mathcal{I}^*$ is *κ-indivisible* if for any κ-coloring σ on $\mathcal{I}^*$, there exists a $\mathcal{L}^*$-substructure $\mathcal{I}_0^*$ of $\mathcal{I}^*$ with $\mathrm{age}_{\mathcal{L}^*}(\mathcal{I}_0^*) = \mathrm{age}_{\mathcal{L}^*}(\mathcal{I}^*)$ such that $\sigma|_{\mathcal{I}_0^*}$ is monocrhomatic.

(ii) We say $\mathcal{I}^*$ is *strongly indivisible* if for any cardinal κ, there exists an $\mathcal{L}^*$-structure $\mathcal{J}^*$ with $\mathrm{age}_{\mathcal{L}^*}(\mathcal{J}^*) = \mathrm{age}_{\mathcal{L}^*}(\mathcal{I}^*)$ such that $\mathcal{J}^*$ is κ-indivisible.

(iii) We say $\mathcal{I}^*$ is *monochromatic* if $\{\langle \eta \rangle \subseteq \mathcal{I}^* : \eta \in \mathcal{I}^*, |\eta| = 1\}$ has only one element up to $\mathcal{L}^*$-isomorphism.

(iv) We say $\mathcal{I}^*$ is *weakly monochromatic* if $\{\langle \eta \rangle \subseteq \mathcal{I}^* : \eta \in \mathcal{I}^*, |\eta| = 1\}$ has only finitely many elements up to $\mathcal{L}^*$-isomorphism.

(v) We say a subset X of $\mathcal{I}^*$ is *cofinal* in $\mathcal{I}^*$ if for any finite subset Z of $\mathcal{I}^*$, there exists an $\mathcal{L}^*$-embedding $f : \mathcal{I}^* \to \mathcal{I}^*$ such that

- $f(\mathcal{I}^*) \cap Z = \emptyset$,
- $\bar{\eta} \sim_{\mathcal{L}^*} \bar{\nu} \Rightarrow \bar{\eta}Z \sim_{\mathcal{L}^*} \bar{\nu}Z$ for all $\bar{\eta}, \bar{\nu} \in f(\mathcal{I}^*)$ with $|\bar{\eta}| = |\bar{\nu}| < \omega$,
- $(f(\mathcal{I}^*) \cap X) \sim^{\mathrm{fin}}_{\mathcal{L}^*} X$.

The class of dividing lines satisfying n-variable theorem

Definition A

We say a quadruple $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfies *n-variable theorem* if

(i) $\mathcal{L}^*$ is a language, $\mathcal{I}^*$ is a locally finite $\mathcal{L}^*$-structure having modeling property, X is a subset of $\mathcal{I}^*$, and $\bar{Y}$ is a finite set of subsets of $\mathcal{I}^*$,

(ii) $\mathcal{I}^*$ is weakly monochromatic,

(iii) X is cofinal in $\mathcal{I}^*$,

(iv) for all $\eta \in \mathcal{I}^*$ with $|\eta| = 1$, there exists $\nu \in X$ with $|\nu| = 1$ such that $\eta \sim_{\mathcal{L}^*} \nu$,

(v) $|Y| > 1$ for each $Y \in \bar{Y}$,

(vi) there exists a disjoint union $W \subseteq \mathcal{I}^*$ of X' and Y' such that
- $X' \sim^{\text{fin}}_{\mathcal{L}^*} X$ and $Y' \sim^{\text{fin}}_{\mathcal{L}^*} Y$ for some $Y \in \bar{Y}$,
- $\{\eta\} \cup X' \sim^{\text{fin}}_{\mathcal{L}^*} X$ for each $\eta \in Y'$.

If $|\bar{Y}| = 1$ and $\bar{Y} = \{Y\}$, then we just say $(\mathcal{L}^*, \mathcal{I}^*, X, Y)$ satisfies n-variable theorem.

The class of dividing lines satisfying n-variable theorem

Definition

For a given quadruple $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfying n-variable theorem, let $D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$ be the class of all first-order theories T such that there do not exist $\varphi(x,y)$ in the language of T and $(a_\eta)_{\eta\in\mathcal{I}^*}$ in a model of T such that

(i) $(a_\eta)_{\eta\in\mathcal{I}^*}$ is $\mathcal{I}^*_{\mathcal{L}^*}$-indiscernible,

(ii) $\{\varphi(x,a_\eta)\}_{\eta\in X}$ is consistent,

(iii) $\{\varphi(x,a_\eta)\}_{\eta\in Y}$ is inconsistent for each $Y\in\bar{Y}$.

Let $\mathfrak{D}_{n\text{-var}}$ be a class of classes of first-order theories such that

$$\mathfrak{D}_{n\text{-var}} := \left\{ D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}} : (\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y}) \text{ satisfies } n\text{-variable theorem} \right\}.$$

Examples of $D \in \mathfrak{D}_{n\text{-var}}$

Let D_{stable} be the class of stable theories. Similarly we define $D_{o-minimal}, D_{simple}, D_{NIP}, D_{NTP_1}, D_{NTP_2}, \ldots$.

Observation

$D_{stable}, D_{simple}, D_{NIP}, D_{NTP_1}, D_{NTP_2}, D_{NATP}, D_{NCTP}, D_{NBTP}, D_{NSOP_3}, D_{NGP}, D_{NPM^{(k)}} \in \mathfrak{D}_{n\text{-var}}$ $(k > 1)$

Most cases can be easily verified from the original definitions. In this talk, we will briefly discuss the cases of NBTP, stability, and NIP, where some additional explanation is needed.

Examples of $D \in \mathfrak{D}_{n\text{-var}}$: NBTP

Lemma [Ahn, K.]

T is NBTP if and only if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

- $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent if X is a strict left-leaning path.
- $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in Y}$ is inconsistent if Y is a strict right-veering path.
- $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in Y}$ is inconsistent if Y a set of direct siblings.

Let $\mathcal{L}_{kr} := \{<_{ds}, <_{ll}, <_{rv}\} \cup \{\Delta^-_{lex}, \Delta^-_{\trianglelefteq}\}$ where $<_{ds}, <_{ll}, <_{rv}$ are binary relations and $\Delta^-_{lex}, \Delta^-_{\trianglelefteq}$ are quaternary relations.

$\eta <_{ds} \nu$ if and only if $\eta = (\eta \wedge \nu)^\frown(i)$ and $\nu = (\eta \wedge \nu)^\frown(j)$ for some $i < j$.

$\eta <_{ll} \nu$ if and only if $\eta = (\eta \wedge \nu)^\frown(i)$ and $\nu \rhd (\eta \wedge \nu)^\frown(j)$ for some $i < j$.

$\eta <_{rv} \nu$ if and only if $\eta \rhd (\eta \wedge \nu)^\frown(i)$ and $\nu = (\eta \wedge \nu)^\frown(j)$ for some $i < j$.

$(\eta_0, \eta_1, \eta_2, \eta_3) \in \Delta^-_{lex}$ if and only if $\eta_0^- \wedge \eta_1^- <_{lex} \eta_2^- \wedge \eta_3^-$.

$(\eta_0, \eta_1, \eta_2, \eta_3) \in \Delta^-_{\trianglelefteq}$ if and only if $\eta_0^- \wedge \eta_1^- \trianglelefteq \eta_2^- \wedge \eta_3^-$.

where

$\eta <_{lex} \nu \Leftrightarrow \eta \trianglelefteq \nu \vee \left[\eta \rhd (\eta \wedge \nu)^\frown(i) \wedge \nu \rhd (\eta \wedge \nu)^\frown(j)\right]$ for some $i < j$,

$\eta^- \lhd \eta$ and $\eta^- \trianglerighteq \mu$ for all $\mu \lhd \eta$.

Proposition [K.]

The $\mathcal{L}_{kr}$-structure on $\omega^{<\omega}$ has modeling property.

Examples of $D \in \mathfrak{D}_{n\text{-var}}$: NBTP

$\mathbb{Q} \times \mathbb{N}$ can be viewed as a linear order under the lexicographic order. Let $\mathcal{T}^{\mathbb{Z}}_{\mathbb{Q}\times\mathbb{N}}$ be the set of maps from end segments of $\mathbb{Q} \times \mathbb{N}$ into $\mathbb{Z}$. Then we can give an $\mathcal{L}_{kr}$-structure on $\mathcal{T}^{\mathbb{Z}}_{\mathbb{Q}\times\mathbb{N}}$ such that $\mathrm{age}_{\mathcal{L}_{kr}}(\mathcal{T}^{\mathbb{Z}}_{\mathbb{Q}\times\mathbb{N}}) = \mathrm{age}_{\mathcal{L}_{kr}}(\omega^{<\omega})$. Then we can find a cofinal X which is a strict left-leaning path. Let Y_0 be an infinite strict right-veering path, and Y_0 an infinite set of direct siblings. Then $(\mathcal{L}_{kr}, \mathcal{T}^{\mathbb{Z}}_{\mathbb{Q}\times\mathbb{N}}, X, (Y_0, Y_1))$ satisfies n-variable theorem, and $D^{\mathcal{L}_{kr}\mathcal{T}^{\mathbb{Z}}_{\mathbb{Q}\times\mathbb{N}}}_{X,(Y_0,Y_1)} = D_{NBTP}$. Thus $D_{NBTP} \in \mathfrak{D}_{n\text{-var}}$.

Examples of $D \in \mathfrak{D}_{n\text{-var}}$: stability and NIP

T is stable if and only if there do not exist $\varphi(\bar{x}, \bar{y})$ and $(a_i)_{i<\omega}$ such that $\{\varphi(a_i, \bar{y})_{i<j}\} \cup \{\neg\varphi(a_i, \bar{y})\}_{i>j}$ is consistent for each $j < \omega$. Thus, stability can be characterized by a consistency-inconsistency configuration. However, this is not a "positive" consistency-inconsistency configuration that we used in the definition of $\mathfrak{D}_{n\text{-var}}$. By "positive", we mean that the consistency and inconsistency patterns are described solely in terms of formulas, without involving their negations.
Recently, Day proved that stability and NIP can also be characterized by positive consistency–inconsistency configurations using colorized Ramsey structures.

Theorem [Day]

T is stable if and only if there are no $\varphi(x, y)$, $(a_\eta)_{\eta \in \mathrm{c}^{<\omega}}$ such that

- $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent if X is a path.
- $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is inconsistent if X is a set of direct siblings,
- $(a_\eta)_{\eta \in \mathrm{c}^{<\omega}}$ is $\mathrm{c}^{<\omega}$-indiscernible,

where c is an infinite 2-colored linear order. That is, a Fraïssé limit of the class of all finite 2-colored linear order. This equivalent condition is called the *colored tree property*.

Examples of $D \in \mathfrak{D}_{n\text{-var}}$: stability and NIP

Theorem [Day]

T is NIP if and only if there are no $\varphi(x, y)$, $(a_{i,j})_{(i,j)\in\omega\times \mathrm{c}}$ such that

- $\{\varphi(x, a_{i,f(i)}\}_{i<\omega}$ is consistent for each $f : \omega \to \mathrm{c}$,
- $\{\varphi(x, a_{i,j})\}_{j\in \mathrm{c}}$ is inconsistent for each $i < \omega$,
- $(a_{i,j})_{(i,j)\in\omega\times \mathrm{c}}$ is $\omega \times \mathrm{c}$-indiscernible,

where c is an infinite 2-colored linear order. That is, a Fraïssé limit of the class of all finite 2-colored linear order. This equivalent condition is called the *colored TP_2*.

Using these results, we can find appropriate quadruples $(\mathcal{L}_0^*, \mathcal{I}_0^*, X_0, Y_0)$, $(\mathcal{L}_1^*, \mathcal{I}_1^*, X_1, Y_1)$ such that $D_{stable} = D^{\mathcal{L}_0^* \mathcal{I}_0^*}_{X_0, Y_0} \in \mathfrak{D}_{n\text{-var}}$, $D_{NIP} = D^{\mathcal{L}_1^* \mathcal{I}_1^*}_{X_1, Y_1} \in \mathfrak{D}_{n\text{-var}}$.

Question

Can this method also be used to characterize NIP_k by a positive consistency–inconsistency configuration?

One-variable theorem

- [Chernikov, Ramsey] If T has TP_1, then there exists $\varphi(\bar{x}, \bar{y})$ witnessing TP_1 such that $|\bar{x}| = 1$.
- [Chernikov] If T has TP_2, then there exists $\varphi(\bar{x}, \bar{y})$ witnessing TP_2 such that $|\bar{x}| = 1$.
- [Ahn, K., Lee] If T has NATP, then there exists $\varphi(\bar{x}, \bar{y})$ witnessing NATP such that $|\bar{x}| = 1$.

We still do not know whether NCTP and NBTP admit a one-variable theorem. However, they do satisfy a property that is somewhat weaker than the one-variable theorem.

n-variable theorem

Definition

By $\dim(\bar{b}/A)$, we mean
$\max\{|\bar{b}^* = (b_0^*, ..., b_{n-1}^*)| : \bar{b}^* \subseteq \bar{b}$ and $b_i^* \notin \mathrm{acl}(\bar{b}^* \setminus \{b_i^*\} \cup A)$ for each $i < n\}$.

n-variable theorem for NCTP [K.]

If T has CTP, then there exists $\varphi(\bar{x}, \bar{y})$ and strongly indiscernible $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

- φ and $(a_\eta)_{\eta \in \omega^{<\omega}}$ witness CTP,
- for each descending comb X, there exists $\bar{b} \models \{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ such that $\dim(\bar{b}/(a_\eta)_{\eta \in X}) = |\bar{b}|$.

Suppose that we have a witness to CTP, namely $\varphi(x, \bar{y})$ and $(a_\eta)_{\eta \in \omega^{<\omega}}$, where $|x| = 1$. Then, for each descending comb X, the set $\{\varphi(x, a_\eta)\}_{\eta \in X}$ has infinitely many solutions. Thus we can find $b \models \{\varphi(x, a_\eta)\}_{\eta \in X}$ such that $\dim(b/(a_\eta)_{\eta \in X}) = 1$. In this sense, n-variable theorem can be regarded as a weaker form of one-variable theorem.

n-variable theorem

n-variable theorem [K.]

Suppose that $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfies n-variable theorem. If $T \notin D^{\mathcal{L}^*, \mathcal{I}^*}_{X,\bar{Y}}$, then there exists $\varphi(\bar{x}, \bar{y})$ and $\mathcal{L}^*$-indiscernible $(a_\eta)_{\eta \in \mathcal{I}^*}$ such that

- φ and $(a_\eta)_{\eta \in \mathcal{I}^*}$ witness $T \notin D^{\mathcal{L}^*, \mathcal{I}^*}_{X,\bar{Y}}$,
- there exists $\bar{b} \models \{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ such that $\dim(\bar{b}/(a_\eta)_{\eta \in X}) = |\bar{b}|$.

Idea of proof. First assume that there is a witness $\varphi(x, \bar{y})$ with $|x| = 1$. Then by using Definition A (vi), we can show that $\{\varphi(x, a_\eta)\}_{\eta \in X}$ has infinitely many solution, which implies the existence of $b \models \{\varphi(x, a_\eta)\}_{\eta \in X}$ such that $\dim(b/(a_\eta)_{\eta \in X}) = 1$. (*)
Now assume that we have a witness $\varphi(\bar{x}, \bar{y})$ with $|\bar{x}| = n$. Note that $\dim(x_0, ..., x_{n-1}/(a_\eta)_{\eta \in X}) = n$ is type-definable. Thus, if there is no $\bar{b} \models \{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ such that $\dim(\bar{b}/(a_\eta)_{\eta \in X}) = n$, then we can find a formula witnessing the failure of $\dim(\bar{x}/(a_\eta)_{\eta \in X}) = n$. Using this formula together with the cofinality of X (Definition A(iii)), we can construct a formula φ' witnessing $T \notin D^{\mathcal{L}^*, \mathcal{I}^*}_{X,\bar{Y}}$, such that the length of its free-variable part is $n - 1$. By (*), we can repeat this process until we obtain a witness of $T \notin D^{\mathcal{L}^*, \mathcal{I}^*}_{X,\bar{Y}}$ satisfying n-variable theorem.

Application of n-variable theorem

Let $\mathcal{L}$ be a language. We say an $\mathcal{L}$-theory T is *geometric* if it eliminates $\exists^\infty$ and acl_T satisfies exchange property. $(a \in \mathrm{acl}_T(bC) \setminus \mathrm{acl}_T(C) \Rightarrow b \in \mathrm{acl}_T(Ca))$
For a geometric theory T and a sufficiently saturated model $M \models T$.

Fact [Berenstein, Vassiliev]

There exists $H \subseteq M$ such that

- $h \notin \mathrm{acl}_T(H \setminus \{h\})$ for each $h \in H$,
- H is dense (i.e., every non-algebraic type has a solution in H),
- H is codense (i.e., for any non-algebraic type $p(x)$ over $A \subseteq M$, there is $m \models p$ such that $m \notin \mathrm{acl}_T(HA)$).

Let P be a new unary predicate symbol, and let $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. Let $H \subseteq M$ be the subset given above. Then the $\mathcal{L}_P$-expansion (M, H) of M is called an *H-structure* of M.

Fact [Berenstein, Vassiliev]

The $\mathcal{L}_P$-theory of H-structure is complete. We denote it by T^{ind}.

Application of n-variable theorem

Let $\mathcal{L}$ be a language, T a geometric $\mathcal{L}$-theory, and M a model of T. Let P be a new predicate symbol and $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. We say an $\mathcal{L}_P$-expansion (M, H) of M is a *lovely pair* if H is dense/codense and $H \prec_{\mathcal{L}} M$. The $\mathcal{L}_P$-theory of lovely pair is also complete. We denote it by T_P.

For each $\mathcal{L}$-formula $\varphi(\bar{x})$, let $P_\varphi(\bar{x})$ be a new predicate and $\mathcal{L}' := \{P_\varphi : \varphi \in \mathcal{L}\}$. Let (M, H) be an H-structure of M. Then H canonically inherits an $\mathcal{L}'$-structure. We call the $\mathcal{L}'$-theory of this structure the *generic trivialization* of M and denote it by T^{gt}.

Fact [Berenstein, Vassiliev]

- T is stable if and only if T^{gt} is stable.
- [+Kaplan] T is simple if and only if T^{gt} is simple.
- T is NIP if and only if T^{gt} is NIP.
- T is NTP_2 if and only if T^{gt} is NTP_2.

Application of n-variable theorem

Let T^* be T^{ind} or T_P.

Fact

- [Berenstein, Vassiliev] T is stable if and only if T^* is stable.
- [Berenstein, Vassiliev] T is simple if and only if T^* is simple.
- [Berenstein, Vassiliev] T is NIP if and only if T^* is NIP.
- [Berenstein, Kim] T is NTP_2 if and only if T^* is NTP_2.
- [Dobrowolski, Kim] T is NTP_1 if and only if T^* is NTP_1.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$ and T be a theory. Then the following are equivalent.

- $T \in D$.
- $T^{ind} \in D$.
- $T_P \in D$.
- $T^{gt} \in D$.

Corollary

- T is NTP_1 if and only if T^{gt} is NTP_1.
- T is NATP if and only if T^{gt} is NATP if and only if T^* is NATP.
- T is NCTP if and only if T^{gt} is NCTP if and only if T^* is NCTP.
- T is NBTP if and only if T^{gt} is NBTP if and only if T^* is NBTP.
- T is NSOP_3 if and only if T^{gt} is NSOP_3 if and only if T^* is NSOP_3.
- T is NGP if and only if T^{gt} is NGP if and only if T^* is NGP.
- T is $\mathrm{NPM}^{(k)}$ if and only if T^{gt} is $\mathrm{NPM}^{(k)}$ if and only if T^* is $\mathrm{NPM}^{(k)}$.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$ and T be a theory. Then the following are equivalent.

- $T \in D$.
- $T^{ind} \in D$.
- $T_P \in D$.
- $T^{gt} \in D$.

Idea of proof. In T^*, it is known that every formula of maximal dimension ($\varphi(\bar{x})$ with $\dim(\varphi(\bar{x})) = |\bar{x}|$) is equivalent, up to a small symmetric difference, to a quantifier-free formula φ' ($\dim(\varphi \triangle \varphi') < |\bar{x}|$). If $T^* \notin D$, then by n-variable theorem, we have a witness of $T^* \notin D$ with maximal dimension. So we can find quantifier-free formula witnessing $T^* \notin D$. By using density and codensity, we can remove P in φ'. This proves $T^* \notin D \Rightarrow T \notin D$. Converse is clear since T^* is an expansion of T.
We can also prove $T^{gt} \notin D \Leftrightarrow T \notin D$ by using the same idea.

Application of n-variable theorem

$D_{o-minimality} \notin \mathfrak{D}_{n\text{-var}}$ (there is a dense/codense expansion of o-minimal structure, which is not o-minimal anymore). Similarly, $D_{distal} \notin \mathfrak{D}_{n\text{-var}}$.

Application of n-variable theorem

Let $\mathbb{F}$ be a field and $\mathcal{L} \supseteq \mathcal{L}_0 := \{0, +, -, \{\lambda\cdot\}_{\lambda\in\mathbb{F}}\}$. Let T be an $\mathcal{L}$-theory such that $T|_{\mathcal{L}_0}$ is the theory of vector space over $\mathbb{F}$. We assume T has quantifier elimination, eliminates quantifier $\exists^\infty$, and $\mathrm{acl}_T = \mathrm{dcl}_T = \mathrm{span}_{\mathbb{F}}$. Fix a subfield K of $\mathbb{F}$. Let G be a new predicate and $\mathcal{L}_G := \mathcal{L} \cup \{G\}$. Let us consider an $\mathcal{L}_G$-theory $T^G \supseteq T$ with the following first-order conditions

- G is dense/codense.
- G is a K-subvector space.
- If $\lambda_0, ..., \lambda_{n-1}$ are K-linearly independence, then for all $g_0, ..., g_{n-1} \in G$,

$$\sum_{i<n} \lambda_i g_i = 0 \Rightarrow \bigwedge_{i<n} g_i = 0$$

.

Fact [Berenstein, Vassiliev, d'Elbee]

- T^G is consistent and complete.
- T^G is NIP if and only T is NIP.
- T^G is stable if and only if T is stable.
- T^G is simple if and only if T is simple.
- T^G is NTP_1 if and only if T is NTP_1.
- T^G is NTP_2 if and only if T is NTP_2.

Application of n-variable theorem

Fact [Ahn, K., Lee, Lee]

T^G is NATP if and only if T is NATP.

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$ and T a theory. Then $T \in D$ if and only if $T^G \in D$.

Corollary

- T^G is NCTP if and only T is NCTP.
- T^G is NBTP if and only if T is NBTP.
- T^G is NSOP_3 if and only if T is NSOP_3.
- T^G is NGP if and only if T is NGP.
- T^G is $\mathrm{NPM}^{(k)}$ if and only if T is $\mathrm{NPM}^{(k)}$.

Idea. T^G is neither an H-structure nor a lovely pair, but it also admits quantifier elimination for formulas of maximal dimension.

Application of n-variable theorem

Let $\mathcal{L} \supseteq \mathcal{L}_{ring}$ and T an $\mathcal{L}$-theory of fields. Let P be a unary predicate and $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. Let ACF_T be an $\mathcal{L}_P$-expansion of ACF satisfying the following.

- P is a subfield of the universe.
- P is a model of T.
- For every n-ary funtion symbol $f \in \mathcal{L} \setminus \mathcal{L}_{ring}$, if $x_0, ..., x_{n-1} \in P$, then $f(x_0, ..., x_{n-1}) \in P$. Else, if some $x_i \notin P$, then we put $f(\bar{x}) = 0$.
- For every n-ary relation symbol $R \in \mathcal{L} \setminus \mathcal{L}_{ring}$, if some $x_i \notin P$, then $\neg R(\bar{x})$.
- The degree of the field extension of the universe over P is infinite. i.e. the universe has infinite dimension as a vector space over P.

Fact [d'Elbee, Kaplan, Neuhauser]

- T is NIP if and only if ACF_T is NIP.
- T is stable if and only if ACF_T is stable.
- T is simple if and only if ACF_T is simple.
- T is NTP_1 if and only if ACF_T is NTP_1.

Fact [Ahn, K., Lee, Lee]

- T is NTP_2 if and only if ACF_T is NTP_2.
- T is NATP if and only if ACF_T is NATP.

Application of n-variable theorem

Theorem [K.]

Let T be a theory and $D \in \mathfrak{D}_{n\text{-var}}$. Then $T \in D$ if and only if $ACF_T \in D$.

Idea. If $a, a' \notin \operatorname{acl}_{\mathcal{L}_{\text{ring}}}(P(\mathbb{M})A)$, then $a \equiv_{P(\mathbb{M})A} a'$. And P is stably embedded, so every $\mathcal{L}_P(\mathbb{M})$-definable set in $P(\mathbb{M})^n$ is defined by an $\mathcal{L}(P(\mathbb{M}))$-formula. Using these facts, we can show that every indiscernible $(a_\eta)_{\eta \in \mathcal{I}^*}$ and $\bar{b}$ with $\dim(\bar{b}/(a_\eta)_{\eta \in \mathcal{I}^*}) = |\bar{b}|$, we may assume that $(a_\eta)_{\eta \in \mathcal{I}^*}$ is indiscernible over $\bar{b}$. It means that, we can find a witness of $\neg D$ with 0 free variable outside P. By stably embeddedness, we can find a witness of $\neg D$ in $\mathcal{L}$. It shows that $ACF_T \notin D \Rightarrow T \notin D$. The other direction is easy.

Question

Can we extend this idea to a 'higher-arity' version?

We can define $(\mathcal{L}^*, \mathcal{I}^*, k, X, \bar{Y})$ in a similar way, except that $X \subseteq (\mathcal{I}^*)^k$ and $Y \subseteq (\mathcal{I}^*)^k$ for each $Y \in \bar{Y}$. Under this setting, we can show T^{gt} preserves $D \in \mathfrak{D}^h_{n\text{-var}}$ assuming acl = dcl.

Conjecture

$D_{NIP_k} \in \mathfrak{D}^h_{n\text{-var}}$ for $k > 1$.

References

[1] Saharon Shelah, Classification Theory, North Holland, Second Edition 1990.

[2] Julia F. Knight, Anand Pillay, and Charles Steinhorn, Definable Sets in Ordered Structures, II, Transactions of the American Mathematical Society, 1986.

[3] Lou van den Dries, Tame Topology and O-Minimal Structures, London Mathematical Society Lecture Note Series, 1998.

[4] JinHoo Ahn and Joonhee Kim, SOP_1, SOP_2, and antichain tree property, Annals of Pure and Applied Logic, 2024.

[5] Scott Mutchnik, On $NSOP_2$ Theories, Journal of the European Mathematical Society, 2025.

[6] James E. Hanson, Bi-invariant Types, Reliably Invariant Types, and the Comb Tree Property, arXiv:2306.08239.

[7] Alex Kruckman and Nicholas Ramsey, A New Kim's Lemma, Model Theory, 2024.

[8] James E. Hanson, A Combinatorial Characterization of Kim's Lemma for Pairs of Bi-Invariant Types, arXiv:2507.21366.

[9] Michele Bailetti, A Walk on the Wild Side: Notions of Maximality in First-Order Theories, Journal of Symbolic Logic, 2025.

[10] Lynn Scow, Vincent Guingona, and Cameron Donnay Hill, Characterizing Model-Theoretic Dividing Lines via Collapse of Generalized Indiscernibles, Annals of Pure and Applied Logic, 2017.

[11] Blaise Boissonneau, Aris Papadopoulos, and Pierre Touchard, Mekler's Construction and Murphy's Law for 2-Nilpotent Groups, Pacific Journal of Mathematics, 2026.

References

[12] JinHoo Ahn and Joonhee Kim, Mekler's Construction and the Preservation of NCTP and NBTP, arXiv:2602.22966.

[13] Gabriel Day, Results on Colored Tree Properties, arXiv:2507.06977.

[14] Artem Chernikov and Nicholas Ramsey, On Model-Theoretic Tree Properties, Journal of Mathematical Logic, 2016.

[15] Artem Chernikov, Theories without the Tree Property of the Second Kind, Annals of Pure and Applied Logic, 2014.

[16] Alexander Berenstein and Evgueni Vassiliev, Generic Trivializations of Geometric Theories, Mathematical Logic Quarterly, 2014.

[17] Alexander Berenstein and Evgueni Vassiliev, Geometric Structures with a Dense Independent Subset, Selecta Mathematica, 2016.

[18] Alexander Berenstein and Hyeung-Joon Kim, Dense Codense Predicates and the NTP_2, Mathematical Logic Quarterly, 2016.

[19] Jan Dobrowolski and Hyeungjoon Kim, A Preservation Theorem for Theories without the Tree Property of the First Kind, Journal of Symbolic Logic, 2019.

[20] Alexander Berenstein, Christian d'Elbée, and Evgueni Vassiliev, Vector Spaces with a Dense-Codense Generic Submodule, Annals of Pure and Applied Logic, 2024.

[21] JinHoo Ahn, Joonhee Kim, Hyoyoon Lee, and Junguk Lee, Preservation of NATP, Journal of Mathematical Logic, 2025.

[22] Christian d'Elbée, Itay Kaplan, and Leor Neuhauser, On Algebraically Closed Fields with a Distinguished Subfield, Israel Journal of Mathematics, 2024.