n-variable theorems for dividing lines of first-order theories characterized by positive consistency-inconsistency configurations.

Joonhee Kim

Korea Institute for Advanced Study

Making Categorical Model Theory Accessible

June 4, 2026, Masaryk University

Dividing lines

One of the main goals of model theory is to classify first-order theories in meaningful ways and to investigate their properties within these classifications. There are many approaches to such classifications.

- Consistency-inconsistency configuration (NTP, NTP_1, NTP_2),
- Existence of a certain independence notion (stable, simple, NSOP_1, rosy),
- Collapsing indiscernible (stable, NIP, k-NIP, NFOP_2),
- Using rank notions (dp-minimal, SU-rank),
- Restricting shapes of definable sets (o-minimal, C-minimal),

  and more.

Dividing lines

For example, stability is characterized by several equivalent conditions, each providing a different viewpoint on the notion.

Fact

TFAE.

- T is stable.
- Every indiscernible sequence $(a_i)_{i<\omega}$ is totally indiscernible (i.e., $a_{i_0}...a_{i_n} \equiv a_{j_0}...a_{j_n}$ for all $i_0, ..., i_n, j_0, ..., j_n \in \omega$).
- There are no $\varphi(x, y)$, $(a_i)_{i<\omega}$, and $(b_i)_{i<\omega}$ such that
$$\models \varphi(a_i, b_j) \quad \Leftrightarrow \quad i < j.$$
- There exists a preindependence relation $\downarrow$ (a ternary invariant relation) satisfying normality, monotonicity, base monotonicity, transitivity, right extension, symmetry, strong finite character, and stationary.

Dividing lines

Such dividing lines are usually defined in terms of the complexity of theories. Roughly speaking, they are determined by *how much complexity* can be encoded into *models* of a given first-order theory using the *language*.

Let $\mathcal{L}$ be a language and T be an $\mathcal{L}$-theory. Let $\mathcal{L}' \supseteq \mathcal{L}$, and let T' be an expansion of T to the language $\mathcal{L}'$. Then one would expect that the *additional expressive power* of $\mathcal{L}'$ allows us to encode more complicated patterns inside the expanded models than the original language $\mathcal{L}$.

Preservation problems

Let D be a dividing line, such as stability, simplicity, NIP, NTP, and so on.
Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Suppose T is captured by D (e.g., T is stable, simple, NIP, NTP, and so on).
Let $\mathcal{L}' \supseteq \mathcal{L}$ and T' be an expansion of T to $\mathcal{L}'$.
Is still T' captured by D?

Dividing lines

Some dividing lines give the theories they capture nice model-theoretic features. A natural question is under what conditions these features are preserved when the expressive power of the language is increased.

Example

$\mathrm{Th}((\mathbb{R}; +, \cdot, 0, 1, <))$ is o-minimal. Let f be a new function symbol, for instance exp, log, and so on. Is $\mathrm{Th}((\mathbb{R}; +, \cdot, 0, 1, <, f))$ still o-minimal?

Fact [Wilkie]

$\mathrm{Th}((\mathbb{R}; +, \cdot, 0, 1, <, f))$ remains o-minimal if $f = \exp$.

Dividing lines

Let T be a complete theory and M its model.

- [Shelah] We say T is *stable* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_i)_{i<\omega}$, and $(b_i)_{i<\omega}$ such that $\models \varphi(a_i, b_j)$ if and only if $i < j$.
- [Shelah] We say T has *NIP* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_i)_{i<\omega}$, and $(b_I)_{I\subseteq\omega}$ such that $\models \varphi(a_i, b_I)$ if and only if $i \in I$.
- [Shelah] We say T is *simple* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that

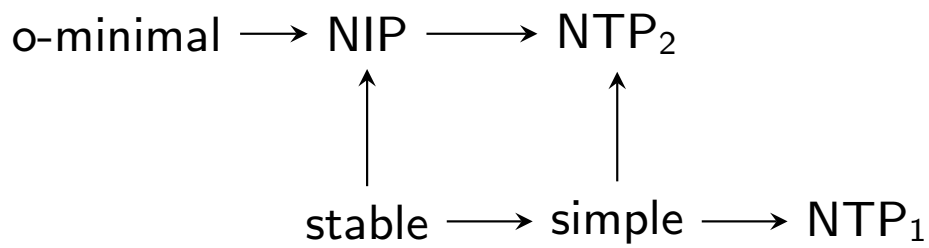
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta\in X}$ is consistent if X is a path in $\omega^{<\omega}$.
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta\in Y}$ is consistent if Y is a set of direct siblings.
- [Knight, Pillay, Steinhorn, van den Dries] We say T is *o-minimal* if it has a definable order $<$ and every definable set in M is a union on finitely many $<$-intervals and finitely many points.
- [Shelah] We say T is *NTP*$_1$ if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that

  $\{\varphi(\bar{x}, a_\eta)\}_{\eta\in X}$ is consistent if and only if X is a path in $\omega^{<\omega}$.
- [Shelah] We say T is *NTP*$_2$ if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_{i,j})_{i,j<\omega}$ such that

  $\{\varphi(\bar{x}, a_{i,f(i)})\}_{j<\omega}$ is consistent for all $f : \omega \to \omega$.
  $\{\varphi(\bar{x}, a_{i,j})\}_{j<\omega}$ is inconsistent for all $i < \omega$.

Dividing lines

The following diagram is well known.



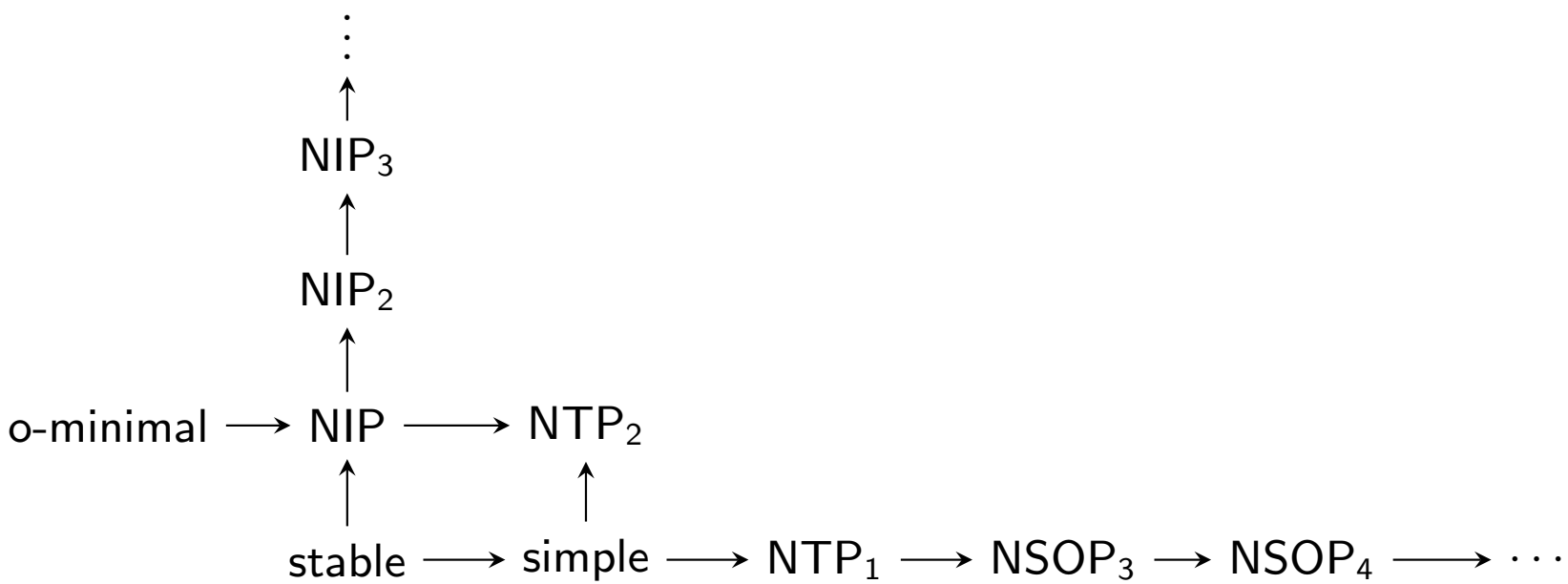
This diagram means that:

- if a theory is stable, then it is NIP and simple,
- if a theory is simple, then it is NTP_1 and NTP_2,
- if a theory is o-minimal, then it is NIP,
- if a theory is NIP, then it is NTP_2.

Dividing lines

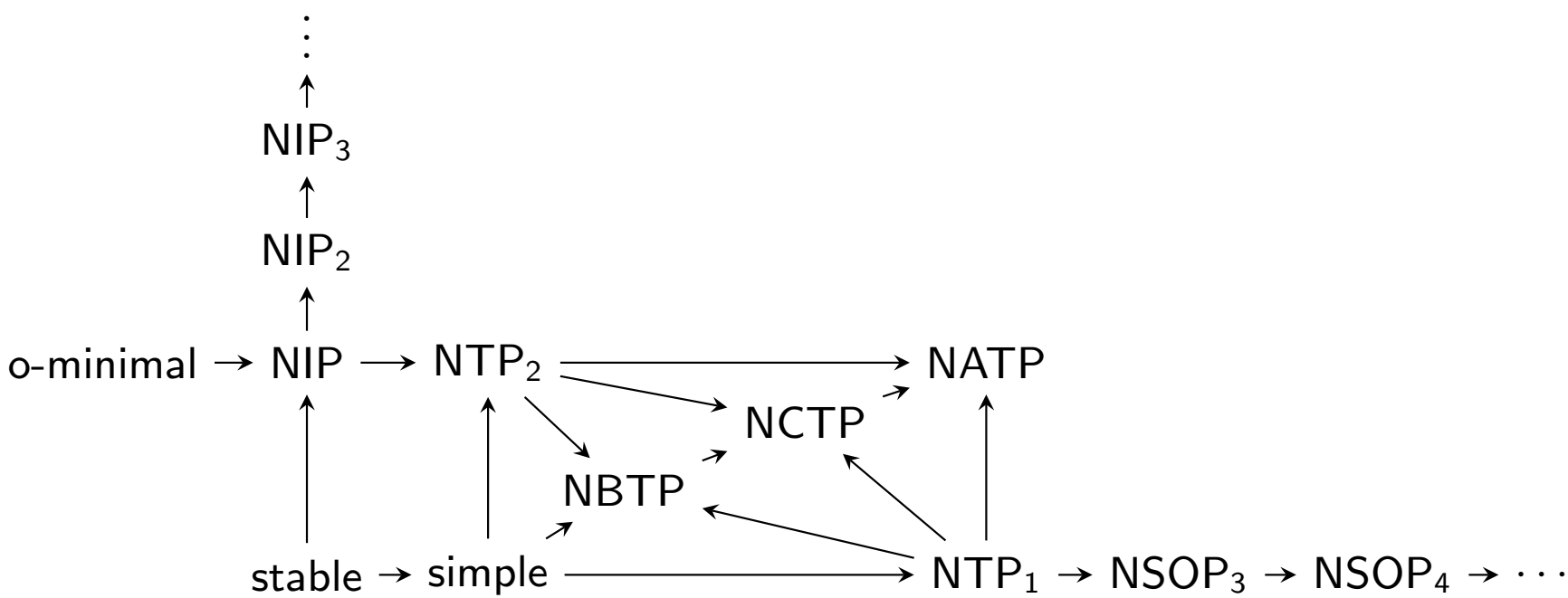
Let $2 < n < \omega$ and $1 < k < \omega$.

- [Shelah] We say T is $NSOP_n$ if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, a_{i+1})$ for all $i < \omega$ and $\{\varphi(\bar{x}_0, \bar{x}_1), ..., \varphi(\bar{x}_{n-2}, \bar{x}_{n-1}), \varphi(\bar{x}_{n-1}, \bar{x}_0)\}$ is inconsistent.
- [Shelah] We say T has NIP_k if there do not exist $\varphi(\bar{x}, \bar{y}_0, ..., \bar{y}_{k-1})$, $(a_i^0, ..., a_i^{k-1})_{i<\omega}$, and $(b_I)_{I \subseteq \omega^k}$ such that $\models \varphi(b_I, a_{i_0}^0, ..., a_{i_{k-1}}^{k-1})$ if and only if $(i_0, ..., i_{k-1}) \in I$.



Dividing lines

- [Ahn, K.] We say T is *NATP* if there are no $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is inconsistent if and only if X is a path in $\omega^{<\omega}$.
- [Mutchnik] We say T is *NCTP* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent if X is a descending comb.
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in Y}$ is inconsistent if Y is a path.
- [Ramsey, Kruckman] We say T is *NBTP* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent if X is a left-leaning path.
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in Y}$ is inconsistent if Y is a right-veering path.



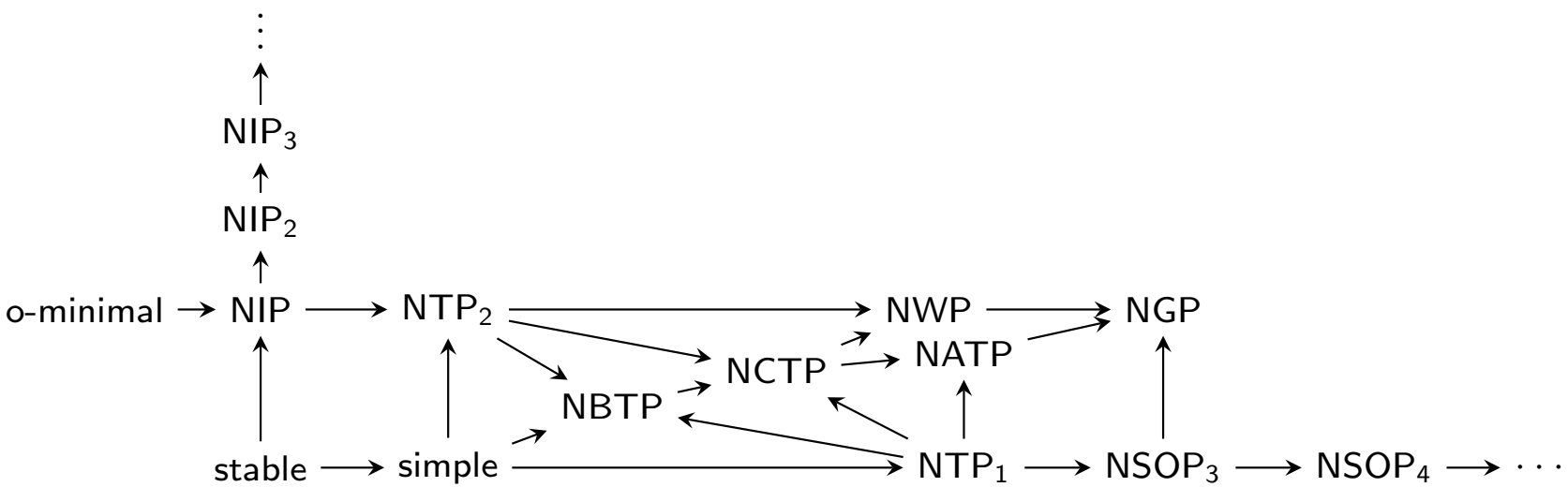
Dividing lines

- [Hanson] For $k < \omega$, we say T is *NWP (having no $(k,1,1)$-weave of depth ω)* if there do not exist $\varphi(\bar{x}, \bar{y})$ and $(a_\eta)_{\eta \in (2^2)^\omega}$ such that

  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent for each finite right-1-comb X,
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is k-inconsistent for each finite up-1-comb X.

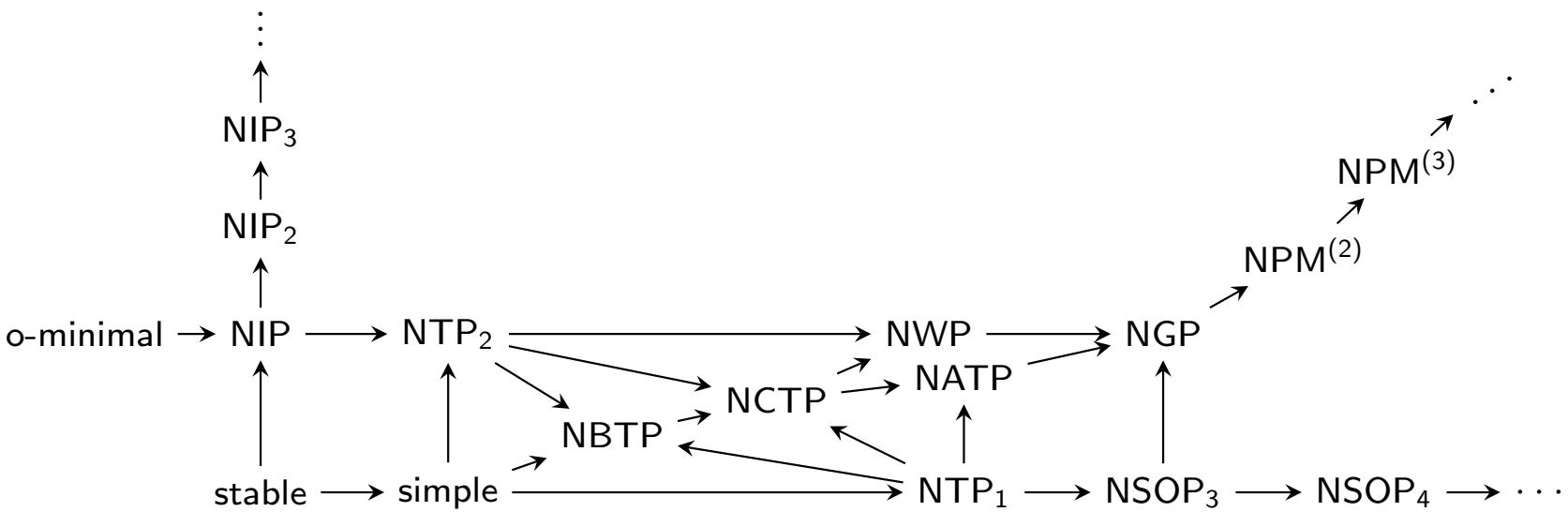
- [Hanson] For $k < \omega$, we say T has an *NGP (having no infinite k-grid)* if there do not exist $\varphi(\bar{x}, \bar{y})$ and $(a_{i,j})_{i,j<\omega}$ such that

  * $\{\varphi(\bar{x}, a_{i,j})\}_{(i,j) \in X}$ is consistent if X satisfies $(i,j), (i',j') \in X \Rightarrow i \leq i' \wedge j \leq j'$ or $i' \leq i \wedge j' \leq j$,
  * $\{\varphi(\bar{x}, a_{i,j})\}_{(i,j) \in X}$ is k-inconsistent if X satisfies $(i,j), (i',j') \in X \Rightarrow i \not\leq i' \vee j \not\leq j'$ and $i' \not\leq i \vee j' \not\leq j$.



Dividing lines

- [Bailetti] For $1 < k < \omega$, we say T is *$NPM^{(k)}$ (having no $PM^{(k)}$)* if there do not exist $\varphi(\bar{x}, \bar{y})$ such that for a generic k-hypergraph (ω, E) on ω, there exists $(b_i)_{i<\omega}$ such that
  - $\{\varphi(\bar{x}, b_i)\}_{i \in X}$ is consistent if $X \subseteq \omega$ is a clique (i.e., $E(i_0, ..., i_{k-1})$ for all $i_0, ..., i_{k-1} \in X$),
  - $\{\varphi(\bar{x}, b_i)\}_{i \in X}$ is inconsistent if $X \subseteq \omega$ contains $i_0, ..., i_{k-1}$ such that $\neg E(i_0, ..., i_{k-1})$.



In fact, all of these dividing lines in the diagram, except for o-minimality and $NSOP_n$ for $n > 3$, can be formulated in a uniform way.

This is one of the three main points of today's talk.

Generalized indiscernible sequences

Definition

Let $\mathcal{L}^*$ be a language and $\mathcal{I}^*$ an $\mathcal{L}^*$-structure. Let $\mathcal{L}$ be another language, T an $\mathcal{L}$-theory, $\mathbb{M}$ a monster model of T, and A a small subset of $\mathbb{M}$.
A family $(a_\eta)_{\eta \in \mathcal{I}^*}$ in $\mathbb{M}$ is said to be *$\mathcal{I}^*$-indiscernible over A* if for all finite tuples $\bar{\eta}, \bar{\nu}$ from $\mathcal{I}^*$,

$$\mathrm{qftp}_{\mathcal{L}^*}(\bar{\eta}) = \mathrm{qftp}_{\mathcal{L}^*}(\bar{\nu}) \quad \Rightarrow \quad \mathrm{tp}_{\mathcal{L}}(a_{\bar{\eta}}) = \mathrm{tp}_{\mathcal{L}}(a_{\bar{\nu}}).$$

Example

Ordinary indiscernible sequences correspond to the case where $\mathcal{L}^* = \{<\}$ and $\mathcal{I}^*$ is a linear order.

$$i_0 < \cdots < i_n \text{ and } j_0 < \cdots < j_n \;\Rightarrow\; a_{i_0} \ldots a_{i_n} \equiv a_{j_0} \ldots a_{j_n}$$

Roughly speaking, the concept of generalized indiscernibility involves two languages $\mathcal{L}$ and $\mathcal{L}^*$, an $\mathcal{L}$-structure $\mathbb{M}$, and an $\mathcal{L}^*$-structure $\mathcal{I}^*$.
A Generalized indiscernible sequence is an $\mathcal{I}^*$-indexed set of parameters from $\mathbb{M}$. If two subtuples of the generalized indiscernible sequence are given, then they have the same $\mathcal{L}$-type in $\mathbb{M}$ when their index have the same $\mathcal{L}^*$-type in $\mathcal{I}^*$.

Generalized indiscernible sequences

Definition

Let $\mathcal{L}$ be an arbitrary language, T an arbitrary complete $\mathcal{L}$-theory, and $M \models T$ a sufficiently saturated model. Let $\mathcal{L}^*$ be a language and $\mathcal{I}^*$ an $\mathcal{L}^*$-structure. We say an $\mathcal{L}^*$-structure $\mathcal{I}^*$ has *modeling property* if for any $(a_i)_{i \in \mathcal{I}^*} \subseteq M$, there exists $\mathcal{I}^*$-indiscernible $(b_i)_{i \in \mathcal{I}^*}$ such that for any $\bar{i} \in \mathcal{I}^*$ and $\varphi(\bar{x}) \in \mathcal{L}$, there exists $\bar{j}$ such that $\text{qftp}_{\mathcal{L}^*}(\bar{i}) = \text{qftp}_{\mathcal{L}^*}(\bar{j})$ and $M \models \varphi(b_{\bar{i}}) \leftrightarrow \varphi(a_{\bar{j}})$.

Example

Let $\mathcal{L}^* := \{<\}$ and $\mathcal{I}^*$ be an infinite linear order. Let $\mathcal{L}$ be another language and $\mathbb{M}$ an $\mathcal{L}$-structure. If $(a_i)_{i \in \mathcal{I}^*}$ is a sequence of elements in $\mathbb{M}$, then there exists $(b_i)_{i \in \mathcal{I}^*} \subseteq \mathbb{M}$ such that

- For $\varphi(x_0, ...x_n) \in \mathcal{L}$ and $i_0 < \cdots < i_n$ in $\mathcal{I}^*$, there exist $j_0 < \cdots < j_n$ such that $\models \varphi(a_{i_0}, ..., a_{i_n}) \leftrightarrow \varphi(b_{j_0}, ..., b_{j_n})$.
- $(b_i)_{i \in \mathcal{I}^*}$ is $\mathcal{I}^*$-indiscernible (usual order-indiscernible).

Generalized indiscernible sequences

Let ω^n be the set of maps from n to ω and $\omega^{<\omega} := \bigcup_{n<\omega} \omega^n$.
We can define binary relations $\lhd, <_{lex}$ and a binary funtion $\wedge$ on $\omega^{<\omega}$.

- $\eta \lhd \nu$ if $\eta \subseteq \nu$.
- $\eta <_{lex} \nu$ if (i) $\eta \lhd \nu$ or
  (ii) $\xi^\frown(i) \trianglelefteq \eta$ and $\xi^\frown(j) \trianglelefteq \nu$ for some $\xi \in \omega^{<\omega}$ and $i < j < \omega$.
- $\eta \wedge \nu = \xi$ if $\xi \trianglelefteq \eta$, $\xi \trianglelefteq \nu$, and ξ is the $\lhd$-largest such element.

Let $\mathcal{L}_0 := \{\lhd, <_{lex}, \wedge\}$, and consider the $\mathcal{L}_0$-structure on $\omega^{<\omega}$ described above.

Generalized indiscernible sequences

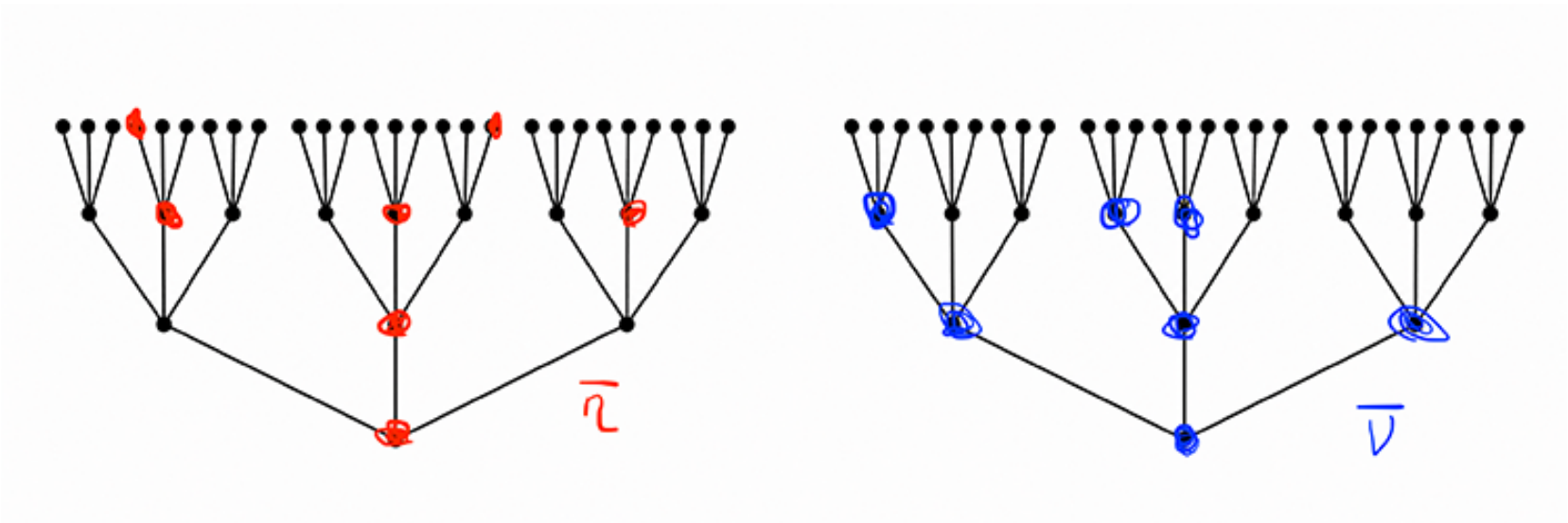
Fact [Takeuchi, Tsuboi]

$\omega^{<\omega}$ has modeling property.

If $(a_\eta)_{\eta\in\omega^{<\omega}} \subseteq \mathbb{M}$ is given, then there exists $\omega^{<\omega}$-indiscernible $(b_\eta)_{\eta\in\omega^{<\omega}}$ such that

for any $\varphi(\bar{x}) \in \mathcal{L}$ and $\bar{\eta} \in \omega^{<\omega}$, there exists $\bar{\nu}$ such that

$$\bar{\eta} \sim_{\mathcal{L}_0} \bar{\nu} \quad \text{and} \quad \models \varphi(a_{\bar{\eta}}) \leftrightarrow \varphi(b_{\bar{\nu}}).$$



The class of dividing lines satisfying n-variable theorem

Let $age_{\mathcal{L}^*}(\mathcal{I}^*)$ be the class of finitely generated $\mathcal{L}^*$-substructures of $\mathcal{I}^*$.

Fact

Let $\mathcal{L}^*$ be a language and $\mathcal{I}^*$, $\mathcal{J}^*$ be $\mathcal{L}^*$-structure. Suppose $\mathcal{I}^*$ and $\mathcal{J}^*$ are locally finite (i.e., every finitely generated substructure is finite).

If $\mathcal{I}^*$ has modeling property and $\mathrm{age}_{\mathcal{L}^*}(\mathcal{I}^*) = \mathrm{age}_{\mathcal{L}^*}(\mathcal{J}^*)$, then $\mathcal{J}^*$ also has modeling property.

It is also well-known that for locally finite $\mathcal{I}^*$, $\mathcal{I}^*$ has modeling property if and only if $\mathrm{age}_{\mathcal{L}^*}(\mathcal{I}^*)$ has Ramsey property.

The class of dividing lines satisfying n-variable theorem

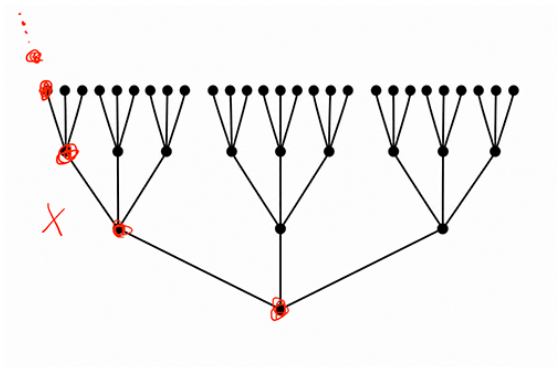
Definition

We say a subset X of $\mathcal{I}^*$ is *cofinal* in $\mathcal{I}^*$ if for any finite subset Z of $\mathcal{I}^*$, there exists an $\mathcal{L}^*$-embedding $f : \mathcal{I}^* \to \mathcal{I}^*$ such that

* $f(\mathcal{I}^*) \cap Z = \emptyset$,
* $\bar{\eta} \sim_{\mathcal{L}^*} \bar{\nu} \Rightarrow \bar{\eta}Z \sim_{\mathcal{L}^*} \bar{\nu}Z$ for all $\bar{\eta}, \bar{\nu} \in f(\mathcal{I}^*)$ with $|\bar{\eta}| = |\bar{\nu}| < \omega$,
* $\mathrm{age}_{\mathcal{L}^*}(f(\mathcal{I}^*) \cap X) = \mathrm{age}_{\mathcal{L}^*}(X)$.

If X is cofinal, then, roughly speaking, X is *sufficiently spread* so that finitely many obstacles do not harm the indiscernibility (by considering an isomorphic image of the whole structure).

Let $X := \{(), (0), (00), (000), (0000), ...\}$. Then for any finitely many $\eta_0, ..., \eta_n$, we can find a sufficiently large natural number m such that $f : \nu \mapsto (0^m)^\frown\nu$ satisfies all conditions above. Thus X is cofinal in $\omega^{<\omega}$.



The class of dividing lines satisfying n-variable theorem

Definition

We say a quadruple $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfies *n-variable theorem* if

(i) $\mathcal{L}^*$ is a language, $\mathcal{I}^*$ is a locally finite $\mathcal{L}^*$-structure having modeling property, X is a subset of $\mathcal{I}^*$, and $\bar{Y}$ is a finite set of subsets of $\mathcal{I}^*$,

(ii) $\{\langle \eta \rangle \subseteq \mathcal{I}^* : \eta \in \mathcal{I}^*, |\eta| = 1\}$ has only finitely many elements up to $\mathcal{L}^*$-isomorphism.

(iii) X is cofinal in $\mathcal{I}^*$,

(iv) there exists a disjoint union $W \subseteq \mathcal{I}^*$ of X' and Y' such that

- $\mathrm{age}(X') = \mathrm{age}(X)$ and $\mathrm{age}(Y') = \mathrm{age}(Y)$ for some $Y \in \bar{Y}$,
- $\mathrm{age}(\{\eta\} \cup X') = \mathrm{age}(X)$ for each $\eta \in Y'$.

If $|\bar{Y}| = 1$ and $\bar{Y} = \{Y\}$, then we just say $(\mathcal{L}^*, \mathcal{I}^*, X, Y)$ satisfies n-variable theorem.

The class of dividing lines satisfying n-variable theorem

Definition

For a given quadruple $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfying n-variable theorem, let $D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$ be the class of all first-order theories T such that there do not exist $\varphi(x,y)$ in the language of T and $(a_\eta)_{\eta\in\mathcal{I}^*}$ in a model of T such that

(i) $(a_\eta)_{\eta\in\mathcal{I}^*}$ is $\mathcal{I}^*_{\mathcal{L}^*}$-indiscernible,

(ii) $\{\varphi(x,a_\eta)\}_{\eta\in X}$ is consistent,

(iii) $\{\varphi(x,a_\eta)\}_{\eta\in Y}$ is inconsistent for each $Y\in\bar{Y}$.

Let $\mathfrak{D}_{n\text{-var}}$ be a class of classes of first-order theories such that

$$\mathfrak{D}_{n\text{-var}} := \left\{ D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}} : (\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y}) \text{ satisfies } n\text{-variable theorem} \right\}.$$

Examples of $D \in \mathfrak{D}_{n\text{-var}}$

Let D_{stable} be the class of stable theories. Similarly we define $D_{o-minimal}, D_{simple}, D_{NIP}, D_{NTP_1}, D_{NTP_2}, \ldots$.

Observation

$D_{stable}, D_{simple}, D_{NIP}, D_{NTP_1}, D_{NTP_2}, D_{NATP}, D_{NCTP}, D_{NBTP}, D_{NSOP_3}, D_{NGP}, D_{NWP}, D_{NPM^{(k)}} \in \mathfrak{D}_{n\text{-var}}$ $(k > 1)$

Examples of $D \in \mathfrak{D}_{n\text{-var}}$: NTP_1

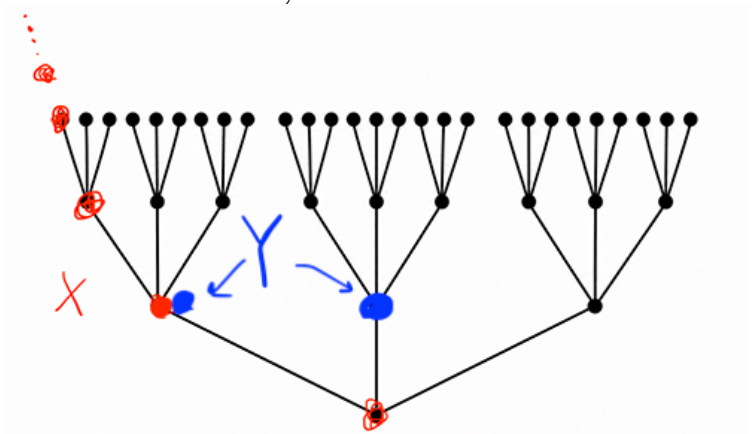
Definition [Shelah]

We say T is NTP_1 if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

$\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent if and only if X is a path in $\omega^{<\omega}$

(i.e. $X := \{\eta_0, ...\}$ such that $\eta_i \lhd \eta_{i+1}$ for all i).

Choose any uncountable cardinal κ with $\mathrm{cf}(\kappa) = \kappa$ and let $X := \{(), (0), (00), ...\}$ and $Y := \{(0), (1)\}$ in $\omega^{<\kappa}$. Then $(\mathcal{L}_0, \omega^{<\kappa}, X, Y)$ satisfies n-variable theorem and $D_{X,Y}^{\mathcal{L}_0, \omega^{<\kappa}} = D_{NTP_1}$. Thus $D_{NTP_1} \in \mathfrak{D}_{n\text{-var}}$.



Similarly, given a dividing line, one may try to find an appropriate structure with Ramsey property, together with a cofinal subset X and a suitable subset Y witnessing the required inconsistency condition.
This is not always possible. $D_{o-minimal}$ does not belong to $\mathfrak{D}_{n\text{-var}}$.

n-variable theorem

As mentioned, having modeling property of $\mathcal{L}^*$-structures is determined by its age. That is, if $\mathrm{age}(\mathcal{I}^*) = \mathrm{age}\mathcal{J}^*$ and $\mathcal{I}^*$ has modeling property, then so does $\mathcal{J}^*$.
But this is no longer true when we ask whether a given $\mathcal{L}^*$-structure admits X and $\bar{Y}$ satisfying the *n*-variable theorem. For example, $\mathrm{age}(\omega^{<\kappa}) = \mathrm{age}(\omega^{<\omega})$ and $\omega^{<\kappa}$ admit X and $\bar{Y}$ satisfying *n*-variable theorem, but $\omega^{<\omega}$ does not admit such subsets satisfying *n*-variable theorem witnessing the same dividing line.

Question

Is there an equivalence relation on $\mathcal{L}^*$-structures that determines whether they admit *n*-variable theorem tuples witnessing the same dividing lines?

If such an equivalence relation exists, then it would be finer than $\mathrm{age}(-) = \mathrm{age}(-)$.

n-variable theorem

Definition

$T \notin D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$ if there are no $\varphi(\bar{x},\bar{y})$ and $\mathcal{I}^*$-indiscernible $(a_\eta)_{\eta\in\mathcal{I}^*}$ such that

(i) $\{\varphi(\bar{x},a_\eta)\}_{\eta\in X}$ is consistent,

(ii) $\{\varphi(\bar{x},a_\eta)\}_{\eta\in Y}$ is inconsistent for each $Y\in\bar{Y}$.

where $\bar{x}$ and $\bar{y}$ are tuples of variables ($\bar{x}$ is of the form $(x_0,\ldots,x_n)$).

In fact, many dividing lines introduced in the literature have already been characterized in terms of consistency and inconsistency patterns, as in the definition above. The main difference in this work is that we make a more refined distinction between the structures carried by the index sets.

n-variable theorem

Once a dividing line has been introduced via a consistency–inconsistency pattern, one of the first questions people tend to ask is the following.

Question

Suppose $T \notin D^{\mathcal{L},*\mathcal{I}^*}_{X,\bar{Y}}$. Is there a witness $\varphi(\bar{x},\bar{y})$ of $T \notin D^{\mathcal{L},*\mathcal{I}^*}_{X,\bar{Y}}$ such that $|\bar{x}| = 1$?

One-variable theorem

- [Chernikov, Ramsey] If T has TP_1, then there exists $\varphi(\bar{x},\bar{y})$ witnessing TP_1 such that $|\bar{x}| = 1$.
- [Chernikov] If T has TP_2, then there exists $\varphi(\bar{x},\bar{y})$ witnessing TP_2 such that $|\bar{x}| = 1$.
- [Ahn, K., Lee] If T has NATP, then there exists $\varphi(\bar{x},\bar{y})$ witnessing NATP such that $|\bar{x}| = 1$.

The second point of the talk: If a dividing line D is a member of $\mathfrak{D}_{n\text{-var}}$ and $T \notin D$, then we can always find a witness $\varphi(\bar{x},\bar{y})$ of $T \notin D$ satisfying *n-variable theorem*, a weaker version of one-variable theorem.

The third point: Although the n-variable theorem is weaker than the one-variable theorem, it is still quite useful, especially when studying preservation problems.

n-variable theorem

Definition (algebraic dimension)

By $\dim(\bar{b}/A)$, we mean
$\max\{|\bar{b}^0 = (b_0^0, ..., b_{n-1}^0)| : \bar{b}^0 \subseteq \bar{b}$ and $b_i^0 \notin \mathrm{acl}(\bar{b}^0 \setminus \{b_i^0\} \cup A)$ for each $i < n\}$.

If $\Gamma(\bar{x})$ is a partial type over a set of parameters A, we define the algebraic dimension of the corresponding type-definable set by

$$\dim(\Gamma(\bar{x})) := \max\{\dim(\bar{b}/A) : \bar{b} \models \Gamma(\bar{x})\}.$$

n-variable theorem [K.]

Suppose that $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfies n-variable theorem. If $T \notin D_{X,\bar{Y}}^{\mathcal{L}^*,\mathcal{I}^*}$, then there exists $\varphi(\bar{x}, \bar{y})$ and $\mathcal{L}^*$-indiscernible $(a_\eta)_{\eta \in \mathcal{I}^*}$ such that

- φ and $(a_\eta)_{\eta \in \mathcal{I}^*}$ witness $T \notin D_{X,\bar{Y}}^{\mathcal{L}^*,\mathcal{I}^*}$,
- $\dim(\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}) = |\bar{x}|$.

Why is it weaker than the one-variable theorem?

If $|\bar{x}| = 1$, then $\dim(\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}) = |\bar{x}|$ if and only if $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ has infinitely many solutions. If $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfies n-variable theorem, then there exists a disjoint union $W \subseteq \mathcal{I}^*$ of X' and Y' such that

(i) $\mathrm{age}(X') = \mathrm{age}(X)$ and $\mathrm{age}(Y') = \mathrm{age}(Y)$ for some $Y \in \bar{Y}$,

(ii) $\mathrm{age}(\{\eta\} \cup X') = \mathrm{age}(X)$ for each $\eta \in Y'$.

If $\dim(\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}) < |\bar{x}| = 1$, then $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is algebraic, so there exists $X'_0 \subseteq X'$ such that $\{\varphi(\mathbb{M}, a_\eta)\}_{\eta \in X'} = \{\varphi(\mathbb{M}, a_\eta)\}_{\eta \in X'_0}$. Thus by (ii), $\{\varphi(\mathbb{M}, a_\eta)\}_{\eta \in X'_0 \cup \{\eta\}} = \{\varphi(\mathbb{M}, a_\eta)\}_{\eta \in X'_0}$ for all $\eta \in Y'$. Thus $\{\varphi(\mathbb{M}, a_\eta)\}_{\eta \in Y'}$ is consistent and it yields a contradiction.

n-variable theorem [K.]

Suppose that $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfies n-variable theorem. If $T \notin D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$, then there exists $\varphi(\bar{x}, \bar{y})$ and $\mathcal{L}^*$-indiscernible $(a_\eta)_{\eta\in\mathcal{I}^*}$ such that

- φ and $(a_\eta)_{\eta\in\mathcal{I}^*}$ witness $T \notin D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$,
- $\dim(\{\varphi(\bar{x}, a_\eta)\}_{\eta\in X}) = |\bar{x}|$.

sketch of proof. If $|x| = 1$, then it already satisfies one-variable theorem, so there is nothing to prove.
Assume that we have a witness $\varphi(\bar{x}, \bar{y})$ with $|\bar{x}| = n$. Note that $\dim(x_0, ..., x_{n-1}/(a_\eta)_{\eta\in X}) = n$ is type-definable. Thus, if there is no $\bar{b} \models \{\varphi(\bar{x}, a_\eta)\}_{\eta\in X}$ such that $\dim(\bar{b}/(a_\eta)_{\eta\in X}) = n$, then we can find a formula witnessing the failure of $\dim(\bar{x}/(a_\eta)_{\eta\in X}) = n$. It witnesses $x_i \in \mathrm{acl}((\bar{x} \setminus \{x_i\}) \cup (a_\eta)_{\eta\in X})$. Since only finitely many parameters a_η occur in the formula, cofinality of X enables us to assume that the whole $(a_\eta)_{\eta\in\mathcal{I}^*}$ is indiscernible over them. Using this, we can find a formula φ' witnessing $T \notin D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$, such that the free-variable part is $\bar{x} \setminus \{x_i\}$. We can repeat this process until we obtain a witness of $T \notin D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$ satisfying n-variable theorem.

Application of n-variable theorem

Let $\mathcal{L}$ be a language. We say an $\mathcal{L}$-theory T is *geometric* if

- it eliminates $\exists^\infty$
  (i.e., for all $\varphi(x, y)$, there exists $\psi_\varphi(y)$ such that $\varphi(x, b)$ is algebraic if and only if $\models \psi_\varphi(b)$),
- acl_T satisfies exchange property.
  (i.e., $a \in \mathrm{acl}_T(bC) \setminus \mathrm{acl}_T(C) \Rightarrow b \in \mathrm{acl}_T(Ca)$)

The following are well-known examples of geometric theories.

- Algebraically closed fields in the ring language.
- Every field with exchange property is geometric.
- Strongly minimal theories.
- SU-rank 1 theories.

Application of n-variable theorem

Fix a geometric theory T and a sufficiently saturated model $M \models T$.

Fact [Berenstein, Vassiliev]

There exists $H \subseteq M$ such that

- $h \notin \mathrm{acl}_T(H \setminus \{h\})$ for each $h \in H$,
- H is dense (i.e., every non-algebraic type has a solution in H),
- H is codense (i.e., for any non-algebraic type $p(x)$ over $A \subseteq M$, there is $m \models p$ such that $m \notin \mathrm{acl}_T(HA)$).

Let P be a new unary predicate symbol, and let $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. Let $H \subseteq M$ be the subset given above. Then the $\mathcal{L}_P$-expansion (M, H) of M is called an *H-structure* of M.

Fact [Berenstein, Vassiliev]

The $\mathcal{L}_P$-theory of H-structure is complete. We denote it by T^{ind}.

Application of n-variable theorem

Let $\mathcal{L}$ be a language, T a geometric $\mathcal{L}$-theory, and M a model of T. Let P be a new predicate symbol and $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. We say an $\mathcal{L}_P$-expansion (M, H) of M is a *lovely pair* if H is dense/codense and $H \prec_{\mathcal{L}} M$. The $\mathcal{L}_P$-theory of lovely pair is also complete. We denote it by T_P.

For each $\mathcal{L}$-formula $\varphi(\bar{x})$, let $P_\varphi(\bar{x})$ be a new predicate and $\mathcal{L}' := \{P_\varphi : \varphi \in \mathcal{L}\}$. Let (M, H) be an H-structure of M. Then H canonically inherits an $\mathcal{L}'$-structure. We call the $\mathcal{L}'$-theory of this structure the *generic trivialization* of M and denote it by T^{gt}.

T^{gt} can be regarded as a Morleyization on an independence dense/codense subset of the universe. And in T^{gt}, acl is trivial (i.e., $\mathrm{acl}(X) = X$ for all X).

Fact [Berenstein, Vassiliev]

- T is stable if and only if T^{gt} is stable.
- [+Kaplan] T is simple if and only if T^{gt} is simple.
- T is NIP if and only if T^{gt} is NIP.
- T is NTP_2 if and only if T^{gt} is NTP_2.

Application of n-variable theorem

Let T^* be T^{ind} or T_P.

Fact

- [Berenstein, Vassiliev] T is stable if and only if T^* is stable.
- [Berenstein, Vassiliev] T is simple if and only if T^* is simple.
- [Berenstein, Vassiliev] T is NIP if and only if T^* is NIP.
- [Berenstein, Kim] T is NTP_2 if and only if T^* is NTP_2.
- [Dobrowolski, Kim] T is NTP_1 if and only if T^* is NTP_1.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$ and T be a theory. Then the following are equivalent.

- $T \in D$.
- $T^{ind} \in D$.
- $T_P \in D$.
- $T^{gt} \in D$.

Corollary

- T is NTP_1 if and only if T^{gt} is NTP_1.
- T is NATP if and only if T^{gt} is NATP if and only if T^* is NATP.
- T is NCTP if and only if T^{gt} is NCTP if and only if T^* is NCTP.
- T is NBTP if and only if T^{gt} is NBTP if and only if T^* is NBTP.
- T is NSOP_3 if and only if T^{gt} is NSOP_3 if and only if T^* is NSOP_3.
- T is NGP if and only if T^{gt} is NGP if and only if T^* is NGP.
- T is NWP if and only if T^{gt} is NWP if and only if T^* is NWP.
- T is $\mathrm{NPM}^{(k)}$ if and only if T^{gt} is $\mathrm{NPM}^{(k)}$ if and only if T^* is $\mathrm{NPM}^{(k)}$.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$ and T be a theory. Then the following are equivalent.

- $T \in D$.
- $T^{ind} \in D$.
- $T_P \in D$.
- $T^{gt} \in D$.

sketch of proof. In T^*, it is known that every formula of maximal dimension ($\varphi(\bar{x})$ with $\dim(\varphi(\bar{x})) = |\bar{x}|$) is equivalent, up to a small symmetric difference, to a quantifier-free formula φ' ($\dim(\varphi \triangle \varphi') < |\bar{x}|$). If $T^* \notin D$, then by n-variable theorem, we have a witness of $T^* \notin D$ with maximal dimension. So we can find quantifier-free formula witnessing $T^* \notin D$. By using density and codensity, we can remove P in φ'. This proves $T^* \notin D \Rightarrow T \notin D$. Converse is clear since T^* is an expansion of T.
We can also prove $T^{gt} \notin D \Leftrightarrow T \notin D$ by using the same idea.

Application of n-variable theorem

$D_{o-minimality} \notin \mathfrak{D}_{n\text{-var}}$ (there is a dense/codense expansion of o-minimal structure, which is not o-minimal anymore).

Application of n-variable theorem

Let $\mathbb{F}$ be a field and $\mathcal{L} \supseteq \mathcal{L}_0 := \{0, +, -, \{\lambda\cdot\}_{\lambda\in\mathbb{F}}\}$. Let T be an $\mathcal{L}$-theory such that $T|_{\mathcal{L}_0}$ is the theory of vector space over $\mathbb{F}$. We assume T has quantifier elimination, eliminates quantifier $\exists^\infty$, and $\mathrm{acl}_T = \mathrm{dcl}_T = \mathrm{span}_{\mathbb{F}}$. Fix a subfield K of $\mathbb{F}$. Let G be a new predicate and $\mathcal{L}_G := \mathcal{L} \cup \{G\}$. Let us consider an $\mathcal{L}_G$-theory $T^G \supseteq T$ with the following first-order conditions

- G is dense/codense.
- G is a K-subvector space.
- If $\lambda_0, ..., \lambda_{n-1}$ are K-linearly independence, then for all $g_0, ..., g_{n-1} \in G$,

$$\sum_{i<n} \lambda_i g_i = 0 \Rightarrow \bigwedge_{i<n} g_i = 0$$

.

Fact [Berenstein, Vassiliev, d'Elbee]

- T^G is consistent and complete.
- T^G is NIP if and only T is NIP.
- T^G is stable if and only if T is stable.
- T^G is simple if and only if T is simple.
- T^G is NTP_1 if and only if T is NTP_1.
- T^G is NTP_2 if and only if T is NTP_2.

Application of n-variable theorem

Fact [Ahn, K., Lee, Lee]

T^G is NATP if and only if T is NATP.

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$ and T a theory. Then $T \in D$ if and only if $T^G \in D$.

Corollary

- T^G is NCTP if and only T is NCTP.
- T^G is NBTP if and only if T is NBTP.
- T^G is NSOP_3 if and only if T is NSOP_3.
- T^G is NGP if and only if T is NGP.
- T^G is NWP if and only if T is NWP.
- T^G is $\mathrm{NPM}^{(k)}$ if and only if T is $\mathrm{NPM}^{(k)}$.

sketch of proof. T^G is neither an H-structure nor a lovely pair, but it also admits quantifier elimination for formulas of maximal dimension.

Application of n-variable theorem

Definition

Let $\mathbb{M} \models T$ and let $D \subseteq M^n$ be definable (possibly with parameters).
We say that D is *stably embedded* if every subset of D^m
definable in $\mathbb{M}$ is already definable in the induced structure on D
with parameters from D.

Example

In the theory ACVF of algebraically closed valued fields, the residue field and the value group are stably embedded.

Application of n-variable theorem

Let $\mathcal{L} \supseteq \mathcal{L}_{ring}$ and T an $\mathcal{L}$-theory of fields. Let P be a unary predicate and $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. Let ACF_T be an $\mathcal{L}_P$-expansion of ACF satisfying the following.

- P is a subfield of the universe.
- P is a model of T.
- For every n-ary funtion symbol $f \in \mathcal{L} \setminus \mathcal{L}_{ring}$, if $x_0, ..., x_{n-1} \in P$, then $f(x_0, ..., x_{n-1}) \in P$. Else, if some $x_i \notin P$, then we put $f(\bar{x}) = 0$.
- For every n-ary relation symbol $R \in \mathcal{L} \setminus \mathcal{L}_{ring}$, if some $x_i \notin P$, then $\neg R(\bar{x})$.
- The degree of the field extension of the universe over P is infinite. i.e. the universe has infinite dimension as a vector space over P.

Fact [d'Elbee, Kaplan, Neuhauser]

- T is NIP if and only if ACF_T is NIP.
- T is stable if and only if ACF_T is stable.
- T is simple if and only if ACF_T is simple.
- T is NTP_1 if and only if ACF_T is NTP_1.

Fact [Ahn, K., Lee, Lee]

- T is NTP_2 if and only if ACF_T is NTP_2.
- T is NATP if and only if ACF_T is NATP.

Application of n-variable theorem

Theorem [K.]

Let T be a theory and $D \in \mathfrak{D}_{n\text{-var}}$. Then $T \in D$ if and only if $ACF_T \in D$.

sketch of proof. If $a, a' \notin \mathrm{acl}_{\mathcal{L}_{\mathrm{ring}}}(P(\mathbb{M})A)$, then $a \equiv_{P(\mathbb{M})A} a'$. And P is stably embedded, so every $\mathcal{L}_P(\mathbb{M})$-definable set in $P(\mathbb{M})^n$ is defined by an $\mathcal{L}(P(\mathbb{M}))$-formula. Using these facts, we can show that every indiscernible $(a_\eta)_{\eta \in \mathcal{I}^*}$ and $\bar{b}$ with $\dim(\bar{b}/(a_\eta)_{\eta \in \mathcal{I}^*}) = |\bar{b}|$, we may assume that $(a_\eta)_{\eta \in \mathcal{I}^*}$ is indiscernible over $\bar{b}$. It means that, we can find a witness of $\neg D$ with 0 free variable outside P. By stably embeddedness, we can find a witness of $\neg D$ in $\mathcal{L}$. It shows that $ACF_T \notin D \Rightarrow T \notin D$. The other direction is easy.

Corollary

- T is NCTP if and only ACF_T is NCTP.
- T is NBTP if and only if ACF_T is NBTP.
- T is NSOP_3 if and only if ACF_T is NSOP_3.
- T is NGP if and only if ACF_T is NGP.
- T is NWP if and only if ACF_T is NWP.
- T is $\mathrm{NPM}^{(k)}$ if and only if ACF_T is $\mathrm{NPM}^{(k)}$.

Higher-arity cases

Can we extend this idea to a 'higher-arity' version?

Definition

We say a quintuple $(\mathcal{L}^*, \mathcal{I}^*, k, X, \bar{Y})$ satisfies *n-variable theorem* if

(i) $\mathcal{L}^*$ is a language, $\mathcal{I}^*$ is a locally finite $\mathcal{L}^*$-structure having modeling property, k is a natural number, X is a subset of $(\mathcal{I}^*)^k$, and $\bar{Y}$ is a finite set of subsets of $(\mathcal{I}^*)^k$,

(ii) $\{\langle \eta \rangle \subseteq \mathcal{I}^* : \eta \in \mathcal{I}^*, |\eta| = k\}$ has only finitely many elements up to $\mathcal{L}^*$-isomorphism.

(iii) X is cofinal in $\mathcal{I}^*$,

(iv) there exists a disjoint union $W \subseteq \mathcal{I}^*$ of X' and Y' such that

- $\mathrm{age}(X') = \mathrm{age}(X)$ and $\mathrm{age}(Y') = \mathrm{age}(Y)$ for some $Y \in \bar{Y}$,
- $\mathrm{age}(\{\eta\} \cup X') = \mathrm{age}(X)$ for each $\eta \in Y'$.

If $|\bar{Y}| = 1$ and $\bar{Y} = \{Y\}$, then we just say $(\mathcal{L}^*, \mathcal{I}^*, k, X, Y)$ satisfies *n*-variable theorem.

Higher-arity cases

Definition

For $(\mathcal{L}^*, \mathcal{I}^*, k, X, \bar{Y})$ satisfying n-variable theorem, let $D^{\mathcal{L}^*,\mathcal{I}^*,k}_{X,\bar{Y}}$ be the class of all first-order theories T such that there do not exist $\varphi(x, y_0, ..., y_{k-1})$ in the language of T and $(a_\eta)_{\eta\in\mathcal{I}^*}$ in a model of T such that

(i) $(a_\eta)_{\eta\in\mathcal{I}^*}$ is $\mathcal{I}^*_{\mathcal{L}^*}$-indiscernible,

(ii) $\{\varphi(x, a_{\eta_0}, ..., a_{\eta_{k-1}})\}_{(\eta_0,...,\eta_{k-1})\in X}$ is consistent,

(iii) $\{\varphi(x, a_{\eta_0}, ..., a_{\eta_{k-1}}))\}_{(\eta_0,...,\eta_{k-1})\in Y}$ is inconsistent for each $Y \in \bar{Y}$.

Let $\mathfrak{D}^h_{n\text{-var}}$ be a class of classes of first-order theories such that

$$\mathfrak{D}^h_{n\text{-var}} := \left\{ D^{\mathcal{L}^*,\mathcal{I}^*,k}_{X,\bar{Y}} : (\mathcal{L}^*, \mathcal{I}^*, k, X, \bar{Y}) \text{ satisfies } n\text{-variable theorem} \right\}.$$

Remark

$\mathfrak{D}_{n\text{-var}} \subseteq \mathfrak{D}^h_{n\text{-var}}$.

Proposition [K.]

$D_{NIP_k} \in \mathfrak{D}^h_{n\text{-var}}$ for each $1 < k < \omega$.

Higher-arity cases

Theorem [K.]

If $T \notin D^{\mathcal{L}^*,\mathcal{I}^*,k}_{X,\bar{Y}}$, then there exists $\varphi(\bar{x}, \bar{y}_0, ..., \bar{y}_k)$ and $\mathcal{L}^*$-indiscernible $(a_\eta)_{\eta\in\mathcal{I}^*}$ such that

- φ and $(a_\eta)_{\eta\in\mathcal{I}^*}$ witness $T \notin D^{\mathcal{L}^*,\mathcal{I}^*,k}_{X,\bar{Y}}$,
- $\dim(\{\varphi(\bar{x}, a_{\eta_0}, ..., a_{\eta_{k-1}})\}_{\eta\in X}) = |\bar{x}|$.

Theorem [K.]

Let $D \in \mathfrak{D}^h_{n\text{-var}}$ and T be a theory. Additionally, we assume that acl $=$ dcl in T. Then the following are equivalent.

- $T \in D$.
- $T^{gt} \in D$.

Corollary

Suppose that acl $=$ dcl in T. Then T is NIP$_k$ if and only if T^{gt} is NIP$_k$.

Questions

Is there a modified version of $\mathfrak{D}_{n\text{-var}}$ for which the one-variable theorem always holds?

Are there any other dividing lines belonging to $\mathfrak{D}_{n\text{-var}}$ or $\mathfrak{D}^h_{n\text{-var}}$?

Are there any other model-theoretic constructions that preserve dividing lines in $\mathfrak{D}_{n\text{-var}}$ and $\mathfrak{D}^h_{n\text{-var}}$?

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