n-variable theorems for dividing lines characterized by positive consistency-inconsistency configurations

Joonhee Kim

Korea Institute for Advanced Study

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University of Notre Dame

① We define a class of dividing lines $\mathfrak{D}_{n\text{-var}}$ in terms of consistency-inconsistency patterns and generalized indiscernibles. And we will show that many dividing lines belong to this class.

② If a dividing line D belongs to $\mathfrak{D}_{n\text{-var}}$ and a first-order theory T does not belongs to D, then we can always find a witness $\varphi(\bar{x}, \bar{y})$ of $T \notin D$ satisfying *n-variable theorem*, a weaker version of one-variable theorem.

③ The one-variable theorem has been used in various ways to address *preservation problems*. Although the n-variable theorem is weaker than one-variable theorem, it still useful when we solve preservation problems.

Dividing lines

One of the main goals of model theory is to classify first-order theories in meaningful ways and to investigate their properties within these classifications. There are many approaches to such classifications.

- Consistency-inconsistency configuration (NTP, NTP_1, NTP_2),
- Existence of a certain independence notion (stable, simple, NSOP_1, rosy),
- Collapsing indiscernible (stable, NIP, k-NIP, NFOP_2),
- Using rank notions (dp-minimal, SU-rank),
- Restricting shapes of definable sets (o-minimal, C-minimal),

  and more.

Dividing lines

For example, stability is characterized by several equivalent conditions, each providing a different viewpoint on the notion.

Fact

TFAE.

- T is stable.
- Every indiscernible sequence $(a_i)_{i<\omega}$ is totally indiscernible (i.e., $a_{i_0}...a_{i_n} \equiv a_{j_0}...a_{j_n}$ for all $i_0, ..., i_n, j_0, ..., j_n \in \omega$).
- There are no $\varphi(x, y)$, $(a_i)_{i<\omega}$, and $(b_i)_{i<\omega}$ such that
$$\models \varphi(a_i, b_j) \quad \Leftrightarrow \quad i < j.$$
- There exists a preindependence relation $\downarrow$ (a ternary invariant relation) satisfying normality, monotonicity, base monotonicity, transitivity, right extension, symmetry, strong finite character, and stationary.

Dividing lines

Such dividing lines are usually defined in terms of the complexity of theories. Roughly speaking, they are determined by *how much complexity* can be encoded into *models* of a given first-order theory using the *language*.

Let $\mathcal{L}$ be a language and T be an $\mathcal{L}$-theory. Let $\mathcal{L}' \supseteq \mathcal{L}$, and let T' be an expansion of T to the language $\mathcal{L}'$. Then one would expect that the *additional expressive power* of $\mathcal{L}'$ allows us to encode more complicated patterns inside the expanded models than the original language $\mathcal{L}$.

Preservation problems

Let D be a dividing line, such as stability, simplicity, NIP, NTP, and so on.
Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Suppose T is captured by D (e.g., T is stable, simple, NIP, NTP, and so on).
Let $\mathcal{L}' \supseteq \mathcal{L}$ and T' be an expansion of T to $\mathcal{L}'$.
Is still T' captured by D?

Dividing lines

Let T be a complete theory and M its model.

- [Shelah] We say T is *stable* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_i)_{i<\omega}$, and $(b_i)_{i<\omega}$ such that $\models \varphi(a_i, b_j)$ if and only if $i < j$.
- [Shelah] We say T has *NIP* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_i)_{i<\omega}$, and $(b_I)_{I\subseteq\omega}$ such that $\models \varphi(a_i, b_I)$ if and only if $i \in I$.
- [Shelah] We say T is *simple* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that

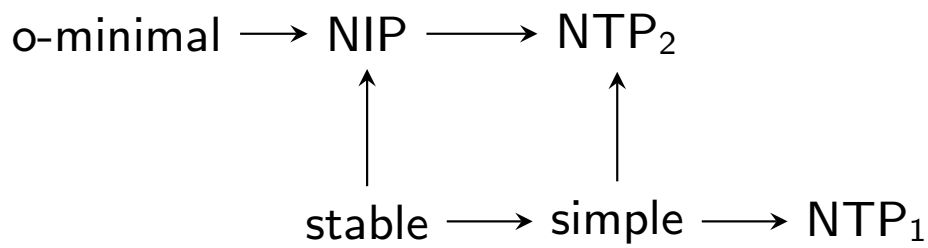
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta\in X}$ is consistent if X is a path in $\omega^{<\omega}$.
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta\in Y}$ is consistent if Y is a set of direct siblings.
- [Knight, Pillay, Steinhorn, van den Dries] We say T is *o-minimal* if it has a definable order $<$ and every definable set in M is a union on finitely many $<$-intervals and finitely many points.
- [Shelah] We say T is *NTP*$_1$ if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that

  $\{\varphi(\bar{x}, a_\eta)\}_{\eta\in X}$ is consistent if and only if X is a path in $\omega^{<\omega}$.
- [Shelah] We say T is *NTP*$_2$ if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_{i,j})_{i,j<\omega}$ such that

  $\{\varphi(\bar{x}, a_{i,f(i)})\}_{j<\omega}$ is consistent for all $f : \omega \to \omega$.
  $\{\varphi(\bar{x}, a_{i,j})\}_{j<\omega}$ is inconsistent for all $i < \omega$.

Dividing lines

The following diagram is well known.



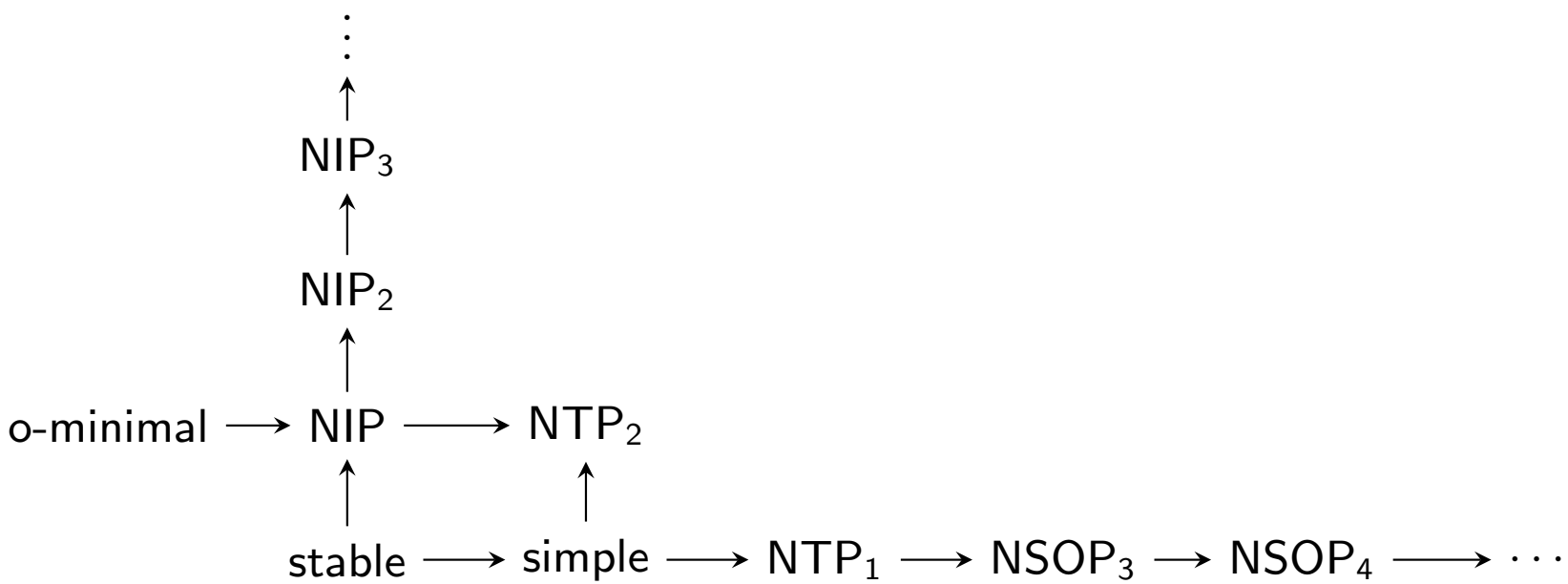
This diagram means that:

- if a theory is stable, then it is NIP and simple,
- if a theory is simple, then it is NTP_1 and NTP_2,
- if a theory is o-minimal, then it is NIP,
- if a theory is NIP, then it is NTP_2.

Dividing lines

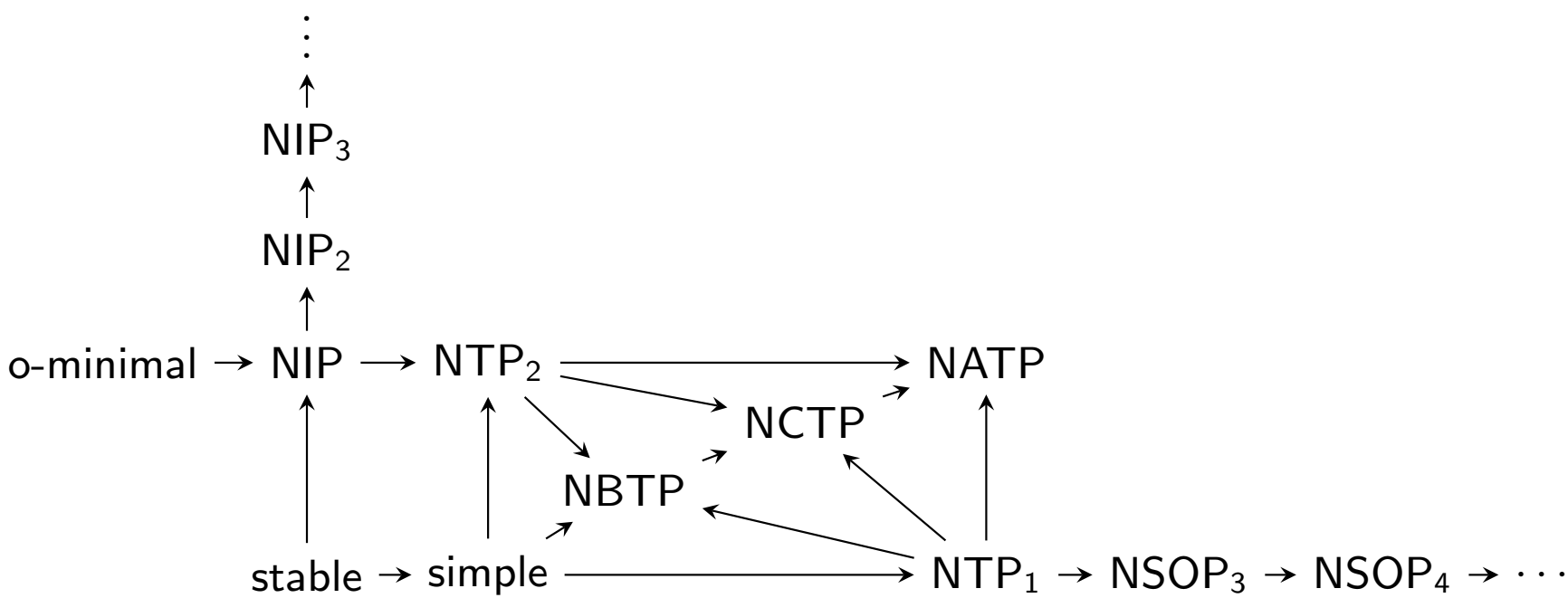
Let $2 < n < \omega$ and $1 < k < \omega$.

- [Shelah] We say T is $NSOP_n$ if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, a_{i+1})$ for all $i < \omega$ and $\{\varphi(\bar{x}_0, \bar{x}_1), ..., \varphi(\bar{x}_{n-2}, \bar{x}_{n-1}), \varphi(\bar{x}_{n-1}, \bar{x}_0)\}$ is inconsistent.
- [Shelah] We say T has NIP_k if there do not exist $\varphi(\bar{x}, \bar{y}_0, ..., \bar{y}_{k-1})$, $(a_i^0, ..., a_i^{k-1})_{i<\omega}$, and $(b_I)_{I \subseteq \omega^k}$ such that $\models \varphi(b_I, a_{i_0}^0, ..., a_{i_{k-1}}^{k-1})$ if and only if $(i_0, ..., i_{k-1}) \in I$.



Dividing lines

- [Ahn, K.] We say T is *NATP* if there are no $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is inconsistent if and only if X is a path in $\omega^{<\omega}$.
- [Mutchnik] We say T is *NCTP* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent if X is a descending comb.
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in Y}$ is inconsistent if Y is a path.
- [Ramsey, Kruckman] We say T is *NBTP* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent if X is a left-leaning path.
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in Y}$ is inconsistent if Y is a right-veering path.



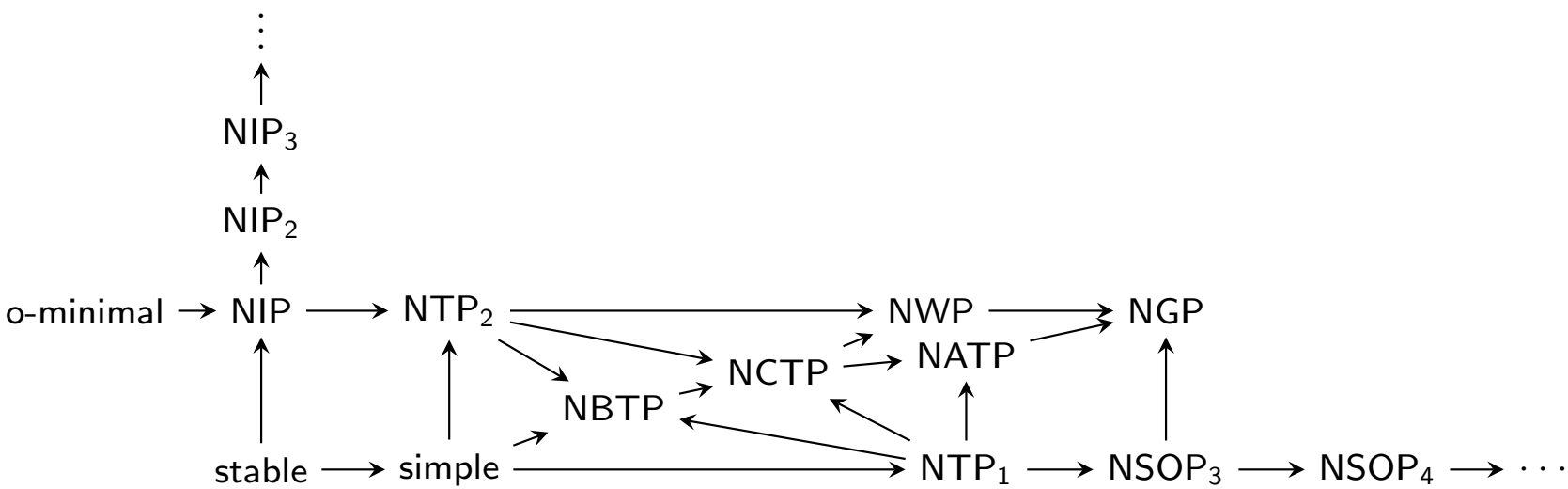
Dividing lines

- [Hanson] For $k < \omega$, we say T is *NWP (having no $(k,1,1)$-weave of depth ω)* if there do not exist $\varphi(\bar{x}, \bar{y})$ and $(a_\eta)_{\eta \in (2^2)^\omega}$ such that

  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent for each finite right-1-comb X,
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is k-inconsistent for each finite up-1-comb X.

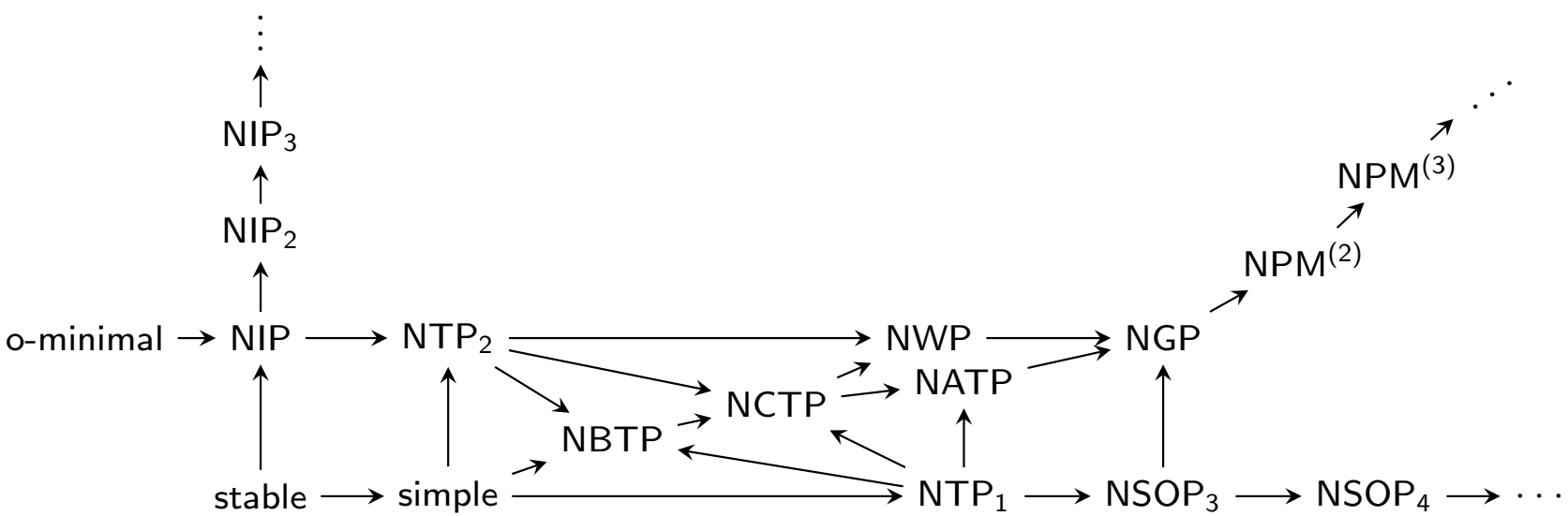
- [Hanson] For $k < \omega$, we say T has an *NGP (having no infinite k-grid)* if there do not exist $\varphi(\bar{x}, \bar{y})$ and $(a_{i,j})_{i,j<\omega}$ such that

  * $\{\varphi(\bar{x}, a_{i,j})\}_{(i,j) \in X}$ is consistent if X satisfies $(i,j),(i',j') \in X \Rightarrow i \leq i' \wedge j \leq j'$ or $i' \leq i \wedge j' \leq j$,
  * $\{\varphi(\bar{x}, a_{i,j})\}_{(i,j) \in X}$ is k-inconsistent if X satisfies $(i,j),(i',j') \in X \Rightarrow i \not\leq i' \vee j \not\leq j'$ and $i' \not\leq i \vee j' \not\leq j$.



Dividing lines

- [Bailetti] For $1 < k < \omega$, we say T is *$NPM^{(k)}$ (having no $PM^{(k)}$)* if there do not exist $\varphi(\bar{x}, \bar{y})$ such that for a generic k-hypergraph (ω, E) on ω, there exists $(b_i)_{i<\omega}$ such that
  - $\{\varphi(\bar{x}, b_i)\}_{i \in X}$ is consistent if $X \subseteq \omega$ is a clique (i.e., $E(i_0, ..., i_{k-1})$ for all $i_0, ..., i_{k-1} \in X$),
  - $\{\varphi(\bar{x}, b_i)\}_{i \in X}$ is inconsistent if $X \subseteq \omega$ contains $i_0, ..., i_{k-1}$ such that $\neg E(i_0, ..., i_{k-1})$.



① We will see that all of these dividing lines in the diagram, except for o-minimality and NSOP_n for $n > 3$, can be formulated in a uniform way.

Generalized indiscernible sequences

Definition

Let $\mathcal{L}^*$ be a language and $\mathcal{I}^*$ an $\mathcal{L}^*$-structure. Let $\mathcal{L}$ be another language, T an $\mathcal{L}$-theory, $\mathbb{M}$ a monster model of T, and A a small subset of $\mathbb{M}$.
A family $(a_\eta)_{\eta \in \mathcal{I}^*}$ in $\mathbb{M}$ is said to be *$\mathcal{I}^*$-indiscernible over A* if for all finite tuples $\bar{\eta}, \bar{\nu}$ from $\mathcal{I}^*$,

$$\mathrm{qftp}_{\mathcal{L}^*}(\bar{\eta}) = \mathrm{qftp}_{\mathcal{L}^*}(\bar{\nu}) \quad \Rightarrow \quad \mathrm{tp}_{\mathcal{L}}(a_{\bar{\eta}}) = \mathrm{tp}_{\mathcal{L}}(a_{\bar{\nu}}).$$

Example

Ordinary indiscernible sequences correspond to the case where $\mathcal{L}^* = \{<\}$ and $\mathcal{I}^*$ is a linear order.

$$i_0 < \cdots < i_n \text{ and } j_0 < \cdots < j_n \;\Rightarrow\; a_{i_0} \ldots a_{i_n} \equiv a_{j_0} \ldots a_{j_n}$$

Roughly speaking, the concept of generalized indiscernibility involves two languages $\mathcal{L}$ and $\mathcal{L}^*$, an $\mathcal{L}$-structure $\mathbb{M}$, and an $\mathcal{L}^*$-structure $\mathcal{I}^*$.
A Generalized indiscernible sequence is an $\mathcal{I}^*$-indexed set of parameters from $\mathbb{M}$. If two subtuples of the generalized indiscernible sequence are given, then they have the same $\mathcal{L}$-type in $\mathbb{M}$ when their index have the same $\mathcal{L}^*$-type in $\mathcal{I}^*$.

Generalized indiscernible sequences

Definition

Let $\mathcal{L}$ be an arbitrary language, T an arbitrary complete $\mathcal{L}$-theory, and $M \models T$ a sufficiently saturated model. Let $\mathcal{L}^*$ be a language and $\mathcal{I}^*$ an $\mathcal{L}^*$-structure. We say an $\mathcal{L}^*$-structure $\mathcal{I}^*$ has *modeling property* if for any $(a_i)_{i \in \mathcal{I}^*} \subseteq M$, there exists $\mathcal{I}^*$-indiscernible $(b_i)_{i \in \mathcal{I}^*}$ such that for any $\bar{i} \in \mathcal{I}^*$ and $\varphi(\bar{x}) \in \mathcal{L}$, there exists $\bar{j}$ such that $\text{qftp}_{\mathcal{L}^*}(\bar{i}) = \text{qftp}_{\mathcal{L}^*}(\bar{j})$ and $M \models \varphi(b_{\bar{i}}) \leftrightarrow \varphi(a_{\bar{j}})$.

Example

Let $\mathcal{L}^* := \{<\}$ and $\mathcal{I}^*$ be an infinite linear order. Let $\mathcal{L}$ be another language and $\mathbb{M}$ an $\mathcal{L}$-structure. If $(a_i)_{i \in \mathcal{I}^*}$ is a sequence of elements in $\mathbb{M}$, then there exists $(b_i)_{i \in \mathcal{I}^*} \subseteq \mathbb{M}$ such that

- For $\varphi(x_0, ...x_n) \in \mathcal{L}$ and $i_0 < \cdots < i_n$ in $\mathcal{I}^*$, there exist $j_0 < \cdots < j_n$ such that $\models \varphi(a_{i_0}, ..., a_{i_n}) \leftrightarrow \varphi(b_{j_0}, ..., b_{j_n})$.
- $(b_i)_{i \in \mathcal{I}^*}$ is $\mathcal{I}^*$-indiscernible (usual order-indiscernible).

Generalized indiscernible sequences

Let ω^n be the set of maps from n to ω and $\omega^{<\omega} := \bigcup_{n<\omega} \omega^n$.
We can define binary relations $\lhd, <_{lex}$ and a binary funtion $\wedge$ on $\omega^{<\omega}$.

- $\eta \lhd \nu$ if $\eta \subseteq \nu$.
- $\eta <_{lex} \nu$ if (i) $\eta \lhd \nu$ or
  (ii) $\xi^\frown(i) \trianglelefteq \eta$ and $\xi^\frown(j) \trianglelefteq \nu$ for some $\xi \in \omega^{<\omega}$ and $i < j < \omega$.
- $\eta \wedge \nu = \xi$ if $\xi \trianglelefteq \eta$, $\xi \trianglelefteq \nu$, and ξ is the $\lhd$-largest such element.

Let $\mathcal{L}_0 := \{\lhd, <_{lex}, \wedge\}$, and consider the $\mathcal{L}_0$-structure on $\omega^{<\omega}$ described above.

Generalized indiscernible sequences

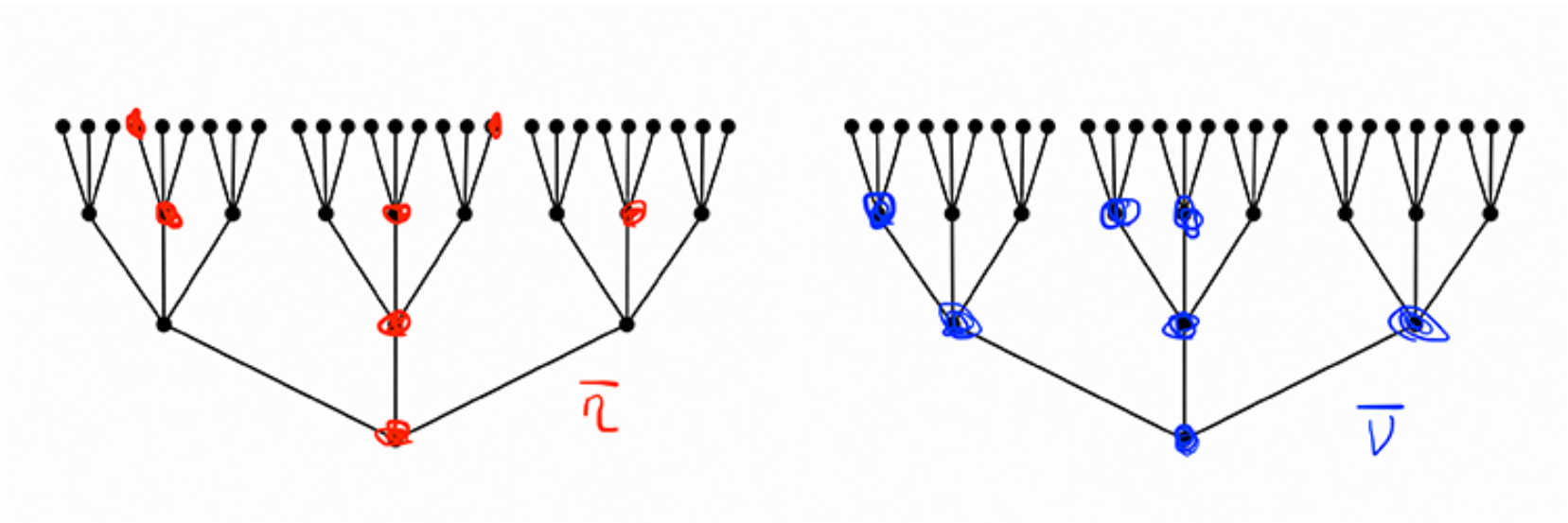
Fact [Takeuchi, Tsuboi] & [Kim, Kim, Scow]

$\omega^{<\omega}$ has modeling property.

If $(a_\eta)_{\eta\in\omega^{<\omega}} \subseteq \mathbb{M}$ is given, then there exists $\omega^{<\omega}$-indiscernible $(b_\eta)_{\eta\in\omega^{<\omega}}$ such that

for any $\varphi(\bar{x}) \in \mathcal{L}$ and $\bar{\eta} \in \omega^{<\omega}$, there exists $\bar{\nu}$ such that

$$\bar{\eta} \sim_{\mathcal{L}_0} \bar{\nu} \quad \text{and} \quad \models \varphi(a_{\bar{\eta}}) \leftrightarrow \varphi(b_{\bar{\nu}}).$$



The class of dividing lines satisfying *n*-variable theorem

Let $age_{\mathcal{L}^*}(\mathcal{I}^*)$ be the class of finitely generated $\mathcal{L}^*$-substructures of $\mathcal{I}^*$.

Fact

Let $\mathcal{L}^*$ be a language and $\mathcal{I}^*$, $\mathcal{J}^*$ be $\mathcal{L}^*$-structure. Suppose $\mathcal{I}^*$ and $\mathcal{J}^*$ are locally finite (i.e., every finitely generated substructure is finite).

If $\mathcal{I}^*$ has modeling property and $\mathrm{age}_{\mathcal{L}^*}(\mathcal{I}^*) = \mathrm{age}_{\mathcal{L}^*}(\mathcal{J}^*)$, then $\mathcal{J}^*$ also has modeling property.

It is also well-known that for locally finite $\mathcal{I}^*$, $\mathcal{I}^*$ has modeling property if and only if $\mathrm{age}_{\mathcal{L}^*}(\mathcal{I}^*)$ has Ramsey property.

The class of dividing lines satisfying n-variable theorem

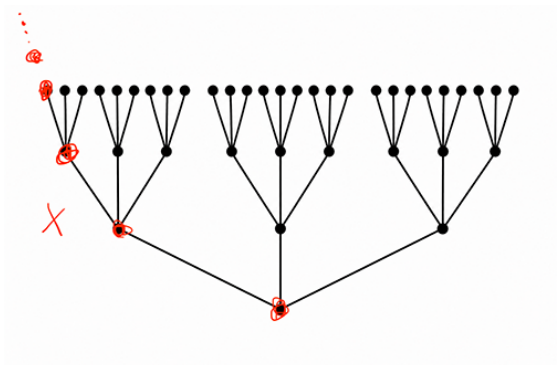
Definition

We say a subset X of $\mathcal{I}^*$ is *cofinal* in $\mathcal{I}^*$ if for any finite subset Z of $\mathcal{I}^*$, there exists an $\mathcal{L}^*$-embedding $f : \mathcal{I}^* \to \mathcal{I}^*$ such that

* $f(\mathcal{I}^*) \cap Z = \emptyset$,
* $\bar{\eta} \sim_{\mathcal{L}^*} \bar{\nu} \Rightarrow \bar{\eta}Z \sim_{\mathcal{L}^*} \bar{\nu}Z$ for all $\bar{\eta}, \bar{\nu} \in f(\mathcal{I}^*)$ with $|\bar{\eta}| = |\bar{\nu}| < \omega$,
* $\mathrm{age}_{\mathcal{L}^*}(f(\mathcal{I}^*) \cap X) = \mathrm{age}_{\mathcal{L}^*}(X)$.

If X is cofinal, then, roughly speaking, X is sufficiently spread so that finitely many obstacles do not harm the indiscernibility (by considering an isomorphic image of the whole structure).

Let $X := \{(), (0), (00), (000), (0000), ...\}$. Then for any finitely many $\eta_0, ..., \eta_n$, we can find a sufficiently large natural number m such that $f : \nu \mapsto (0^m)^\frown\nu$ satisfies all conditions above. Thus X is cofinal in $\omega^{<\omega}$.



The class of dividing lines satisfying n-variable theorem

Definition

We say a quadruple $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfies *n-variable theorem* if

(i) $\mathcal{L}^*$ is a language, $\mathcal{I}^*$ is a locally finite $\mathcal{L}^*$-structure having modeling property, X is a subset of $\mathcal{I}^*$, and $\bar{Y}$ is a finite set of subsets of $\mathcal{I}^*$,

(ii) $\{\langle \eta \rangle \subseteq \mathcal{I}^* : \eta \in \mathcal{I}^*, |\eta| = 1\}$ has only finitely many elements up to $\mathcal{L}^*$-isomorphism.

(iii) X is cofinal in $\mathcal{I}^*$,

(iv) there exists a disjoint union $W \subseteq \mathcal{I}^*$ of X' and Y' such that

- $\mathrm{age}(X') = \mathrm{age}(X)$ and $\mathrm{age}(Y') = \mathrm{age}(Y)$ for some $Y \in \bar{Y}$,
- $\mathrm{age}(\{\eta\} \cup X') = \mathrm{age}(X)$ for each $\eta \in Y'$.

If $|\bar{Y}| = 1$ and $\bar{Y} = \{Y\}$, then we just say $(\mathcal{L}^*, \mathcal{I}^*, X, Y)$ satisfies n-variable theorem.

The class of dividing lines satisfying n-variable theorem

Definition

For a given quadruple $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfying n-variable theorem, let $D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$ be the class of all first-order theories T such that there do not exist $\varphi(x,y)$ in the language of T and $(a_\eta)_{\eta\in\mathcal{I}^*}$ in a model of T such that

(i) $(a_\eta)_{\eta\in\mathcal{I}^*}$ is $\mathcal{I}^*_{\mathcal{L}^*}$-indiscernible,

(ii) $\{\varphi(x,a_\eta)\}_{\eta\in X}$ is consistent,

(iii) $\{\varphi(x,a_\eta)\}_{\eta\in Y}$ is inconsistent for each $Y\in\bar{Y}$.

Let $\mathfrak{D}_{n\text{-var}}$ be a class of classes of first-order theories such that

$$\mathfrak{D}_{n\text{-var}} := \left\{ D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}} : (\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y}) \text{ satisfies } n\text{-variable theorem} \right\}.$$

Examples of $D \in \mathfrak{D}_{n\text{-var}}$

Let D_{stable} be the class of stable theories. Similarly we define $D_{o-minimal}, D_{simple}, D_{NIP}, D_{NTP_1}, D_{NTP_2}, \ldots$.

Observation

$D_{stable}, D_{simple}, D_{NIP}, D_{NTP_1}, D_{NTP_2}, D_{NATP}, D_{NCTP}, D_{NBTP}, D_{NSOP_3}, D_{NGP}, D_{NWP}, D_{NPM^{(k)}} \in \mathfrak{D}_{n\text{-var}}$ $(k > 1)$

Examples of $D \in \mathfrak{D}_{n\text{-var}}$: NTP_1

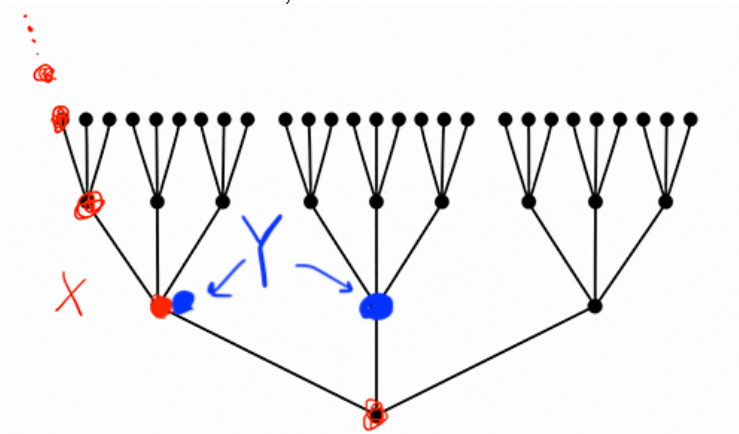
Definition [Shelah]

We say T is NTP_1 if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

$\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent if and only if X is a path in $\omega^{<\omega}$

(i.e. $X := \{\eta_0, ...\}$ such that $\eta_i \lhd \eta_{i+1}$ for all i).

Choose any uncountable cardinal κ with $\mathrm{cf}(\kappa) = \kappa$ and let $X := \{(), (0), (00), ...\}$ and $Y := \{(0), (1)\}$ in $\omega^{<\kappa}$. Then $(\mathcal{L}_0, \omega^{<\kappa}, X, Y)$ satisfies n-variable theorem and $D_{X,Y}^{\mathcal{L}_0, \omega^{<\kappa}} = D_{NTP_1}$. Thus $D_{NTP_1} \in \mathfrak{D}_{n\text{-var}}$.



Similarly, given a dividing line, one may try to find an appropriate structure with Ramsey property, together with a cofinal subset X and a suitable subset Y witnessing the required inconsistency condition.
This is not always possible. $D_{o-minimal} \notin \mathfrak{D}_{n\text{-var}}$.

As mentioned, having modeling property of $\mathcal{L}^*$-structures is determined by its age. That is, if $\mathrm{age}(\mathcal{I}^*) = \mathrm{age}(\mathcal{J}^*)$ and $\mathcal{I}^*$ has modeling property, then so does $\mathcal{J}^*$.
But this is no longer true when we ask whether a given $\mathcal{L}^*$-structure admits X and $\bar{Y}$ satisfying the *n*-variable theorem. For example, $\mathrm{age}(\omega^{<\kappa}) = \mathrm{age}(\omega^{<\omega})$ and $\omega^{<\kappa}$ admit X and $\bar{Y}$ satisfying *n*-variable theorem, but $\omega^{<\omega}$ does not admit such subsets satisfying *n*-variable theorem witnessing the same dividing line.

Question

Is there an equivalence relation on $\mathcal{L}^*$-structures that determines whether they admit *n*-variable theorem tuples witnessing the same dividing lines?

If such an equivalence relation exists, then it would be finer than $\mathrm{age}(-) = \mathrm{age}(-)$.

n-variable theorem

Definition

Suppose $T \notin D^{\mathcal{L}^*;\mathcal{I}^*}_{X,\bar{Y}}$. Then there exist $\varphi(\bar{x},\bar{y})$ and $\mathcal{I}^*$-indiscernible $(a_\eta)_{\eta\in\mathcal{I}^*}$ such that

(i) $\{\varphi(\bar{x},a_\eta)\}_{\eta\in X}$ is consistent,

(ii) $\{\varphi(\bar{x},a_\eta)\}_{\eta\in Y}$ is inconsistent for each $Y \in \bar{Y}$.

where $\bar{x}$ and $\bar{y}$ are tuples of variables ($\bar{x}$ is of the form $(x_0,\ldots,x_n)$).
We say φ *witnesses* $T \notin D^{\mathcal{L}^*;\mathcal{I}^*}_{X,\bar{Y}}$ with $(a_\eta)_{\eta\in\mathcal{I}^*}$).

In fact, many dividing lines introduced in the literature have already been characterized in terms of consistency and inconsistency patterns, as in the definition above. The main difference in this work is that we make a more refined distinction between the structures carried by the index sets.

n-variable theorem

Once a dividing line has been introduced via a consistency–inconsistency pattern, one of the first questions people tend to ask is the following.

Question

Suppose $T \notin D^{\mathcal{L}^*,\mathcal{I}^*}_{\bar{X},\bar{Y}}$. Is there a witness $\varphi(\bar{x},\bar{y})$ of $T \notin D^{\mathcal{L}^*,\mathcal{I}^*}_{\bar{X},\bar{Y}}$ such that $|\bar{x}| = 1$?

One-variable theorem

- [Chernikov, Ramsey] If T has TP_1, then there exists $\varphi(\bar{x},\bar{y})$ witnessing TP_1 such that $|\bar{x}| = 1$.
- [Chernikov] If T has TP_2, then there exists $\varphi(\bar{x},\bar{y})$ witnessing TP_2 such that $|\bar{x}| = 1$.
- [Ahn, K., Lee] If T has NATP, then there exists $\varphi(\bar{x},\bar{y})$ witnessing NATP such that $|\bar{x}| = 1$.

② If a dividing line D is a member of $\mathfrak{D}_{n\text{-var}}$ and $T \notin D$, then we can always find a witness $\varphi(\bar{x},\bar{y})$ of $T \notin D$ satisfying *n-variable theorem*, a weaker version of one-variable theorem.

Definition (algebraic dimension)

By $\dim(\bar{b}/A)$, we mean
$\max\{|\bar{b}^0 = (b^0_0, ..., b^0_{n-1})| : \bar{b}^0 \subseteq \bar{b}$ and $b^0_i \notin \mathrm{acl}(\bar{b}^0 \setminus \{b^0_i\} \cup A)$ for each $i < n\}$.

If $\Gamma(\bar{x})$ is a partial type over a set of parameters A, we define the algebraic dimension of the corresponding type-definable set by

$$\dim(\Gamma(\bar{x})) := \max\{\dim(\bar{b}/A) : \bar{b} \models \Gamma(\bar{x})\}.$$

n-variable theorem [K.]

Suppose that $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfies n-variable theorem. If $T \notin D^{\mathcal{L}^*, \mathcal{I}^*}_{X,\bar{Y}}$, then there exists $\varphi(\bar{x}, \bar{y})$ and $\mathcal{L}^*$-indiscernible $(a_\eta)_{\eta \in \mathcal{I}^*}$ such that

- φ and $(a_\eta)_{\eta \in \mathcal{I}^*}$ witness $T \notin D^{\mathcal{L}^*, \mathcal{I}^*}_{X,\bar{Y}}$,
- $\dim(\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}) = |\bar{x}|$.

Why is it weaker than the one-variable theorem?

If $|\bar{x}| = 1$, then $\dim(\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}) = |\bar{x}|$ if and only if $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ has infinitely many solutions. If $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfies n-variable theorem, then there exists a disjoint union $W \subseteq \mathcal{I}^*$ of X' and Y' such that

(i) $\mathrm{age}(X') = \mathrm{age}(X)$ and $\mathrm{age}(Y') = \mathrm{age}(Y)$ for some $Y \in \bar{Y}$,

(ii) $\mathrm{age}(\{\eta\} \cup X') = \mathrm{age}(X)$ for each $\eta \in Y'$.

If $\dim(\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}) < |\bar{x}| = 1$, then $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is algebraic, so there exists $X'_0 \subseteq X'$ such that $\{\varphi(\mathbb{M}, a_\eta)\}_{\eta \in X'} = \{\varphi(\mathbb{M}, a_\eta)\}_{\eta \in X'_0}$. Thus by (ii), $\{\varphi(\mathbb{M}, a_\eta)\}_{\eta \in X'_0 \cup \{\eta\}} = \{\varphi(\mathbb{M}, a_\eta)\}_{\eta \in X'_0}$ for all $\eta \in Y'$. Thus $\{\varphi(\mathbb{M}, a_\eta)\}_{\eta \in Y'}$ is consistent and it yields a contradiction.

n-variable theorem [K.]

Suppose that $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfies n-variable theorem. If $T \notin D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$, then there exists $\varphi(\bar{x}, \bar{y})$ and $\mathcal{L}^*$-indiscernible $(a_\eta)_{\eta\in\mathcal{I}^*}$ such that

- φ and $(a_\eta)_{\eta\in\mathcal{I}^*}$ witness $T \notin D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$,
- $\dim(\{\varphi(\bar{x}, a_\eta)\}_{\eta\in X}) = |\bar{x}|$.

sketch of proof. If $|x| = 1$, then it already satisfies one-variable theorem, so there is nothing to prove.
Assume that we have a witness $\varphi(\bar{x}, \bar{y})$ with $|\bar{x}| = n$. Note that $\dim(x_0, ..., x_{n-1}/(a_\eta)_{\eta\in X}) = n$ is type-definable. Thus, if there is no $\bar{b} \models \{\varphi(\bar{x}, a_\eta)\}_{\eta\in X}$ such that $\dim(\bar{b}/(a_\eta)_{\eta\in X}) = n$, then we can find a formula witnessing the failure of $\dim(\bar{x}/(a_\eta)_{\eta\in X}) = n$. It witnesses $x_i \in \mathrm{acl}((\bar{x} \setminus \{x_i\}) \cup (a_\eta)_{\eta\in X})$. Since only finitely many parameters a_η occur in the formula, cofinality of X enables us to assume that the whole $(a_\eta)_{\eta\in\mathcal{I}^*}$ is indiscernible over them. Using this, we can find a formula φ' witnessing $T \notin D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$, such that the free-variable part is $\bar{x} \setminus \{x_i\}$. We can repeat this process until we obtain a witness of $T \notin D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$ satisfying n-variable theorem.

Application of n-variable theorem

③ The one-variable theorem has been used in various ways to address preservation problems. Although the n-variable theorem is weaker than one-variable theorem, it is still useful when we solve preservation problems.

Let $\mathcal{L}$ be a language. We say an $\mathcal{L}$-theory T is *geometric* if

- it eliminates $\exists^\infty$
  (i.e., for all $\varphi(x, y)$, there exists $\psi_\varphi(y)$ such that $\varphi(x, b)$ is algebraic if and only if $\models \psi_\varphi(b)$),
- acl_T satisfies exchange property.
  (i.e., $a \in \mathrm{acl}_T(bC) \setminus \mathrm{acl}_T(C) \Rightarrow b \in \mathrm{acl}_T(Ca)$)

Application of n-variable theorem

Fix a geometric theory T and a sufficiently saturated model $M \models T$.

Fact [Berenstein, Vassiliev]

There exists $H \subseteq M$ such that

- $h \notin \mathrm{acl}_T(H \setminus \{h\})$ for each $h \in H$,
- H is dense (i.e., every non-algebraic type over $A \subseteq M$ has a solution in H),
- H is codense (i.e., every non-algebraic type over $A \subseteq M$ has a solution outside $\mathrm{acl}_T(HA)$).

H is said to be independent and dense/codense.

Let P be a new unary predicate symbol, and let $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. Let $H \subseteq M$ be a independent dense/codense subset. Then the $\mathcal{L}_P$-expansion (M, H) of M is called an *H-structure* of M.

Fact [Berenstein, Vassiliev]

The $\mathcal{L}_P$-theory of H-structure is complete. We denote it by T^{ind}.

Application of n-variable theorem

Let $\mathcal{L}$ be a language, T a geometric $\mathcal{L}$-theory, and M a model of T. Let P be a new predicate symbol and $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. We say an $\mathcal{L}_P$-expansion (M, H) of M is a *lovely pair* if H is dense/codense and $H \prec_{\mathcal{L}} M$.

Fact [Berenstein, Vassiliev]

If $H \subseteq M$ is independent dense/codense, then $\mathrm{acl}(H)$ is still dense/codense and $\mathrm{acl}(H) \prec_{\mathcal{L}} M$. The $\mathcal{L}_P$-theory of lovely pair is complete. We denote it by T_P.

For each $\mathcal{L}$-formula $\varphi(\bar{x})$, let $P_\varphi(\bar{x})$ be a new predicate and $\mathcal{L}' := \{P_\varphi : \varphi \in \mathcal{L}\}$. Let (M, H) be an H-structure of M. Then H canonically inherits an $\mathcal{L}'$-structure. We call the $\mathcal{L}'$-theory of this structure the *generic trivialization* of M and denote it by T^{gt}.

T^{gt} can be regarded as a Morleyization on an independence dense/codense subset.

$\mathrm{acl}_{T^{gt}}$ is trivial (i.e., $\mathrm{acl}_{T^{gt}}(x) = x$ for all $x \in M$).

Application of n-variable theorem

Fact [Berenstein, Vassiliev]

- T is stable if and only if T^{gt} is stable.
- [+Kaplan] T is simple if and only if T^{gt} is simple.
- T is NIP if and only if T^{gt} is NIP.
- T is NTP_2 if and only if T^{gt} is NTP_2.

Fact [Berenstein, Vassiliev]

- T is stable if and only if T^{ind} is stable if and only if T_P is stable.
- T is simple if and only if T^{ind} is simple if and only if T_P is simple.
- T is NIP if and only if T^{ind} is NIP if and only if T_P is NIP.

Fact [Berenstein, Kim]

T is NTP_2 if and only if T^{ind} is NTP_2 if and only if T_P is NTP_2.

Fact [Dobrowolski, Kim]

T is NTP_1 if and only if T^{ind} is NTP_1 if and only if T_P is NTP_1.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$ and T be a theory. Then the following are equivalent.

- $T \in D$.
- $T^{gt} \in D$.

Corollary

- T is NTP_1 if and only if T^{gt} is NTP_1.
- T is NATP if and only if T^{gt} is NATP.
- T is NCTP if and only if T^{gt} is NCTP.
- T is NBTP if and only if T^{gt} is NBTP.
- T is NSOP_3 if and only if T^{gt} is NSOP_3.
- T is NGP if and only if T^{gt} is NGP .
- T is NWP if and only if T^{gt} is NWP .
- T is $\mathrm{NPM}^{(k)}$ if and only if T^{gt} is $\mathrm{NPM}^{(k)}$.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$ and T be a theory. Then the following are equivalent.

- $T \in D$.
- $T^{ind} \in D$.
- $T_P \in D$.

Corollary

- T is NCTP if and only if T^{ind} is NCTP if and only if T_P is NCTP.
- T is NBTP if and only if T^{ind} is NBTP if and only if T_P is NBTP.
- T is NSOP_3 if and only if T^{ind} is NSOP_3 if and only if T_P is NSOP_3.
- T is NGP if and only if T^{ind} is NGP if and only if T_P is NGP.
- T is NWP if and only if T^{ind} is NWP if and only if T_P is NWP.
- T is $\mathrm{NPM}^{(k)}$ if and only if T^{ind} is $\mathrm{NPM}^{(k)}$ if and only if T_P is $\mathrm{NPM}^{(k)}$.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$ and T be a theory. Then the following are equivalent.

- $T \in D$.
- $T^{ind} \in D$.
- $T_P \in D$.

sketch of proof. If $\mathrm{tp}_{\mathcal{L}}(\bar{a}) = \mathrm{tp}_{\mathcal{L}}(\bar{b})$ and $\mathrm{tp}_{\{P\}}(\bar{a}) = \mathrm{tp}_{\{P\}}(\bar{b})$ for P-independent $\bar{a}$ and $\bar{b}$, (i.e., $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{a}_{P(\bar{a})} P(M)$ and $\bar{b} \mathop{\smile\!\!\!\!\!\!|}^{a}_{P(\bar{b})} P(M)$), then $\mathrm{tp}_{\mathcal{L}_P}(\bar{a}) = \mathrm{tp}_{\mathcal{L}_P}(\bar{b})$. If $T^{ind} \notin D$, then there exists $\mathcal{L}_G$-formula $\varphi(x, y)$ witnessing $T^{ind} \notin D$. By n-variable theorem, we may assume that realizations of $\varphi(x, a_\eta)$ are P-independent and have the same $\{P\}$-type, so we can find $\mathcal{L}$-formula ψ such that $\varphi(\mathbb{M}, a_\eta) \subseteq \psi(\mathbb{M}, a_\eta)$ and $\psi(\mathbb{M}, a_\eta) \setminus \varphi(\mathbb{M}, a_\eta)$ is 'small' with respect to algebraic dimension. This $\mathcal{L}$-formula ψ witnesses $T \notin D$. The converse is trivial.
The same argument works for T_P.

Application of n-variable theorem

$D_{o-minimality} \notin \mathfrak{D}_{n\text{-var}}$ (there is a dense/codense expansion of o-minimal structure, which is not o-minimal).

Application of n-variable theorem

Let $\mathbb{F}$ be a field and $\mathcal{L} \supseteq \mathcal{L}_0 := \{0, +, -, \{\lambda\cdot\}_{\lambda\in\mathbb{F}}\}$. Let T be an $\mathcal{L}$-theory such that $T|_{\mathcal{L}_0}$ is the theory of vector space over $\mathbb{F}$. We assume T has quantifier elimination, eliminates quantifier $\exists^\infty$, and $\mathrm{acl}_T = \mathrm{dcl}_T = \mathrm{span}_{\mathbb{F}}$.

Fix a subring R of $\mathbb{F}$ and let $\hat{R} := \mathrm{Frac}(R)$ and $\mathcal{L}_{R\text{-mod}} := \{0, +, -, \{r\cdot\}_{r\in R}\}$

Let G be a new unary predicate, $P_\phi(\bar{x})$ a new predicate for each $\phi(\bar{x}) \in \mathcal{L}_{R\text{-mod}}$, and $\mathcal{L}_G := \mathcal{L} \cup \{G\} \cup \{P_\phi(\bar{x}) : \phi(\bar{x}) \in \mathcal{L}_{R\text{-mod}}\}$.

Let us consider an $\mathcal{L}_G$-theory $T_R^G \supseteq T$ with the following first-order conditions

- G is dense/codense.
- G is a R-submodule.
- For $\bar{g} \in G$, $P_\phi(\bar{g})$ holds if $\phi(\bar{g})$ holds in the R-module G.
- If $\lambda_0, ..., \lambda_{n-1}$ are $\hat{R}$-linearly independent, then for all $g_0, ..., g_{n-1} \in G$,

$$\sum_{i<n} \lambda_i g_i = 0 \Rightarrow \bigwedge_{i<n} g_i = 0.$$

Remark

If $R \subsetneq \mathbb{F}$, then $(V, G) \models T_R^G$ is not a lovely pair, not a H-structure.

Application of n-variable theorem

Fact [Berenstein, Vassiliev, d'Elbee]

- T_R^G is NIP if and only T is NIP.
- T_R^G is stable if and only if T is stable.
- T_R^G is simple if and only if T is simple.
- T_R^G is NTP_1 if and only if T is NTP_1.
- T_R^G is NTP_2 if and only if T is NTP_2.

Fact [Ahn, K., Lee, Lee]

T_R^G is NATP if and only if T is NATP.

Application of n-variable theorem

Theorem [K.]

Additionally assume that R is a subfield of $\mathbb{F}$. If $D \in \mathfrak{D}_{n\text{-var}}$, then $T \in D$ if and only if $T_R^G \in D$.

Corollary

- T is NCTP if and only T_R^G is NCTP.
- T is NBTP if and only if T_R^G is NBTP.
- T is NSOP_3 if and only if T_R^G is NSOP_3.
- T is NGP if and only if T_R^G is NGP.
- T is NWP if and only if T_R^G is NWP.
- T is $\mathrm{NPM}^{(k)}$ if and only if T_R^G is $\mathrm{NPM}^{(k)}$.

Application of n-variable theorem

Theorem [K.]

Additionally assume that R is a subfield of $\mathbb{F}$. If $D \in \mathfrak{D}_{n\text{-var}}$, then $T \in D$ if and only if $T_R^G \in D$.

sketch of proof. It uses a quantifier-elimination result similar to those for T^{ind} and T_P, namely, if $\mathrm{tp}_{\mathcal{L}}(\bar{a}) = \mathrm{tp}_{\mathcal{L}}(\bar{b})$, $\mathrm{tp}_{\{G\}}(\bar{a}) = \mathrm{tp}_{\{G\}}(\bar{b})$, $\mathrm{tp}_{R\text{-mod}}(\bar{a}) = \mathrm{tp}_{R\text{-mod}}(\bar{b})$, and $\bar{a}$, $\bar{b}$ are G-independent, then $\mathrm{tp}_{\mathcal{L}_G}(\bar{a}) = \mathrm{tp}_{\mathcal{L}_G}(\bar{b})$. Since we assume R is a field, we can drop the condition $\mathrm{tp}_{R\text{-mod}}(\bar{a}) = \mathrm{tp}_{R\text{-mod}}(\bar{b})$. The rest of the argument is the same as in the cases of T^{ind} and T_P.

Application of n-variable theorem

Definition

Let $\mathbb{M} \models T$ and let $D \subseteq M^n$ be definable (possibly with parameters).
We say that D is *stably embedded* if every subset of D^m
definable in $\mathbb{M}$ is already definable in the induced structure on D
with parameters from D.

Application of n-variable theorem

Let $\mathcal{L} \supseteq \mathcal{L}_{ring}$ and T an $\mathcal{L}$-theory of fields. Let P be a unary predicate and $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. Let ACF_T be an $\mathcal{L}_P$-expansion of ACF satisfying the following.

- P is a subfield of the universe.
- P is a model of T.
- For every n-ary funtion symbol $f \in \mathcal{L} \setminus \mathcal{L}_{ring}$, if $x_0, ..., x_{n-1} \in P$, then $f(x_0, ..., x_{n-1}) \in P$. Else, if some $x_i \notin P$, then we put $f(\bar{x}) = 0$.
- For every n-ary relation symbol $R \in \mathcal{L} \setminus \mathcal{L}_{ring}$, if some $x_i \notin P$, then $\neg R(\bar{x})$.
- The degree of the field extension of the universe over P is infinite. i.e. the universe has infinite dimension as a vector space over P.

Application of n-variable theorem

Fact [d'Elbee, Kaplan, Neuhauser]

- T is NIP if and only if ACF_T is NIP.
- T is stable if and only if ACF_T is stable.
- T is simple if and only if ACF_T is simple.
- T is NTP_1 if and only if ACF_T is NTP_1.

Fact [Ahn, K., Lee, Lee]

- T is NTP_2 if and only if ACF_T is NTP_2.
- T is NATP if and only if ACF_T is NATP.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$. Then $T \in D$ if and only if $ACF_T \in D$.

Corollary

- T is NCTP if and only ACF_T is NCTP.
- T is NBTP if and only if ACF_T is NBTP.
- T is NSOP_3 if and only if ACF_T is NSOP_3.
- T is NGP if and only if ACF_T is NGP.
- T is NWP if and only if ACF_T is NWP.
- T is $\text{NPM}^{(k)}$ if and only if ACF_T is $\text{NPM}^{(k)}$.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$. Then $T \in D$ if and only if $ACF_T \in D$.

sketch of proof. If $a, a' \notin \mathrm{acl}_{\mathcal{L}_{\mathrm{ring}}}(P(\mathbb{M})A)$, then $a \equiv_{P(\mathbb{M})A} a'$. Using these facts, we may assume that $(a_\eta)_{\eta \in \mathcal{I}^*}$ is indiscernible over $\bar{b}$, if $(a_\eta)_{\eta \in \mathcal{I}^*}$ is indiscernible and $\bar{b}$ with $\dim(\bar{b}/P(\mathbb{M})(a_\eta)_{\eta \in \mathcal{I}^*}) = |\bar{b}|$. So if an $\mathcal{L}_P$-formula $\varphi(\bar{x}, y)$ witnesses $ACF_T \notin D$, then we may assume that the realizations of $\varphi(\bar{x}, a_\eta)$ are in $P(\mathbb{M})^{|\bar{x}|}$.
Since P is stably embedded, every $\mathcal{L}_P(\mathbb{M})$-definable set in $P(\mathbb{M})^n$ is defined by an $\mathcal{L}(P(\mathbb{M}))$-formula. Thus we can find a witness of $\neg D$ in $\mathcal{L}$. It shows that $ACF_T \notin D \Rightarrow T \notin D$. The other direction is clear.

Application of n-variable theorem

Let $\mathcal{L} \supseteq \mathcal{L}_{ring}$ and T an $\mathcal{L}$-theory of fields with characteristic 0.

Definition

We say T is *algebraically bounded* if $a \in \mathrm{acl}(KB) \Leftrightarrow \mathrm{trdeg}(a|K(B)) = 0$ for any $K \prec K^* \models T$, $a \in K^*$, and $B \subseteq K^*$.

Let δ be a unary function symbol and $\mathcal{L}_\delta := \mathcal{L} \cup \{\delta\}$. For algebraically bounded T, let T_g^δ be an expansion of T to $\mathcal{L}_\delta$ such that

- $\delta(xy) = \delta(x)y + x\delta(y)$,
- for any $\mathcal{L}(K)$-definable set $X \subseteq K^{r+1}$ for $K \models T_g^\delta$, if the projection of X onto the first r coordinates has algebraic dimension r, then there is $a \in K$ such that $(a, \delta(a), ..., \delta^r(a)) \in X$.

Application of n-variable theorem

Fact [Fornasiero, Terzo]

- T is stable if and only if T_g^δ is stable.
- T is NIP if and only if T_g^δ is NIP.
- T is distal if and only if T_g^δ is distal.
- T is simple if and only if T_g^δ is simple.

Fact [Sánchez, Mohamed]

T is NSOP_1 if and only if T_g^δ is NSOP_1.

Fact [Kaplan, Kesting]

- T is NTP_2 if and only if T_g^δ is NTP_2.
- T is NATP if and only if T_g^δ is NATP.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$. Then $T \in D$ if and only if $T_g^\delta \in D$.

Corollary

- T is NCTP if and only T_g^δ is NCTP.
- T is NBTP if and only if T_g^δ is NBTP.
- T is NSOP_3 if and only if T_g^δ is NSOP_3.
- T is NGP if and only if AT_g^δ is NGP.
- T is NWP if and only if T_g^δ is NWP.
- T is $\mathrm{NPM}^{(k)}$ if and only if T_g^δ is $\mathrm{NPM}^{(k)}$.

Higher arity

Definition

We say a quintuple $(\mathcal{L}^*, \mathcal{I}^*, k, X, \bar{Y})$ satisfies *n-variable theorem* if

(i) $\mathcal{L}^*$ is a language, $\mathcal{I}^*$ is a locally finite $\mathcal{L}^*$-structure having modeling property, k is a natural number, X is a subset of $(\mathcal{I}^*)^k$, and $\bar{Y}$ is a finite set of subsets of $(\mathcal{I}^*)^k$,

(ii) $\{\langle \eta \rangle \subseteq \mathcal{I}^* : \eta \in \mathcal{I}^*, |\eta| = k\}$ has only finitely many elements up to $\mathcal{L}^*$-isomorphism.

(iii) X is cofinal in $\mathcal{I}^*$,

(iv) there exists a disjoint union $W \subseteq \mathcal{I}^*$ of X' and Y' such that

- $\mathrm{age}(X') = \mathrm{age}(X)$ and $\mathrm{age}(Y') = \mathrm{age}(Y)$ for some $Y \in \bar{Y}$,
- $\mathrm{age}(\{\eta\} \cup X') = \mathrm{age}(X)$ for each $\eta \in Y'$.

If $|\bar{Y}| = 1$ and $\bar{Y} = \{Y\}$, then we say $(\mathcal{L}^*, \mathcal{I}^*, k, X, Y)$ satisfies n-variable theorem.

Higher arity

Definition

For $(\mathcal{L}^*, \mathcal{I}^*, k, X, \bar{Y})$ satisfying n-variable theorem, let $D^{\mathcal{L}^*,\mathcal{I}^*,k}_{X,\bar{Y}}$ be the class of all first-order theories T such that there do not exist $\varphi(x, y_0, ..., y_{k-1})$ in the language of T and $(a_\eta)_{\eta\in\mathcal{I}^*}$ in a model of T such that

(i) $(a_\eta)_{\eta\in\mathcal{I}^*}$ is $\mathcal{I}^*_{\mathcal{L}^*}$-indiscernible,

(ii) $\{\varphi(x, a_{\eta_0}, ..., a_{\eta_{k-1}})\}_{(\eta_0,...,\eta_{k-1})\in X}$ is consistent,

(iii) $\{\varphi(x, a_{\eta_0}, ..., a_{\eta_{k-1}}))\}_{(\eta_0,...,\eta_{k-1})\in Y}$ is inconsistent for each $Y \in \bar{Y}$.

Let $\mathfrak{D}^h_{n\text{-var}}$ be a class of classes of first-order theories such that

$$\mathfrak{D}^h_{n\text{-var}} := \left\{ D^{\mathcal{L}^*,\mathcal{I}^*,k}_{X,\bar{Y}} : (\mathcal{L}^*, \mathcal{I}^*, k, X, \bar{Y}) \text{ satisfies } n\text{-variable theorem} \right\}.$$

Remark

$\mathfrak{D}_{n\text{-var}} \subseteq \mathfrak{D}^h_{n\text{-var}}$.

Higher arity

Theorem [K.]

If $T \notin D^{\mathcal{L}^*,\mathcal{I}^*,k}_{X,\bar{Y}}$, then there exists $\varphi(\bar{x}, \bar{y}_0, ..., \bar{y}_k)$ and $\mathcal{L}^*$-indiscernible $(a_\eta)_{\eta \in \mathcal{I}^*}$ such that

- φ and $(a_\eta)_{\eta \in \mathcal{I}^*}$ witness $T \notin D^{\mathcal{L}^*,\mathcal{I}^*,k}_{X,\bar{Y}}$,
- $\dim(\{\varphi(\bar{x}, a_{\eta_0}, ..., a_{\eta_{k-1}})\}_{\eta \in X}) = |\bar{x}|$.

Higher arity

Fact [Day]

- NIP can be characterized by coloured version of TP_2.
- Stability can be characterized by coloured version of TP.

Higher arity

Let $\mathcal{L}_{tp2} := \{<, E\}$ and consider the $\mathcal{L}_{tp2}$-structure on $\omega \times \omega$ such that

(i) $<$ is the lexicographic order on $\omega \times \omega$,

(ii) $E((i,j),(i',j'))$ if $i = i'$.

Then the $\mathcal{L}_{tp2}$-structure on $\omega \times \omega$ has modeling property.
Let $X := \{(i,0) : i < \omega\}$ and $Y := \{(0,i) : i < \omega\}$. Then $(\mathcal{L}_{tp2}, \omega \times \omega, X, Y)$ satisfies n-variable theorem and $D_{NTP_2} = D_{X,Y}^{\mathcal{L}_{tp2}, \omega \times \omega}$.

Let C be a unary relation symbol and $\mathcal{L}_{tp2}^C := \mathcal{L}_{tp2} \cup \{C\}$. Then

$$\{A : A \text{ is a finite } \mathcal{L}_{tp2}^C\text{-structure}, A|_{\mathcal{L}_{tp2}} \in \mathrm{age}_{\mathcal{L}_{tp2}}(\omega \times \omega)\}$$

forms a Fraïssé class having Ramsey property. So its Fraïssé limit $\mathcal{I}_{tp2}^C$ has modeling property. $\mathcal{I}_{tp2}^C$ can be regarded as a randomly coloured $\mathcal{I}_{tp2}$, where $\mathcal{I}_{tp2}$ is a Fraisse limit of $\mathrm{age}_{\mathcal{L}_{tp2}}(\omega \times \omega)$. If we expand the $\mathcal{L}_{tp2}$-structure on $\omega \times \omega$ above to $\mathcal{L}_{tp2}^C$ by $(i,j) \in C$ if and only if j is even, then $\mathrm{age}_{\mathcal{L}_{tp2}^C}(\omega \times \omega) = \mathrm{age}_{\mathcal{L}_{tp2}^C}(\mathcal{I}_{tp2}^C)$. If $X := \{(i,0) : i < \omega\}$ and $Y := \{(0,i) : i < \omega\}$, then $(\mathcal{L}_{tp2}^C, \omega \times \omega, X, Y)$ satisfies n-variable theorem and $D_{NIP} = D_{X,Y}^{\mathcal{L}_{tp2}^C, \omega \times \omega}$.

Higher arity

Let ds, ll, rv be binary relation symbols, $\Delta^-_{<_{lex}}, \Delta^-_{\trianglelefteq}, \Delta^-_{I}$ quaternary relation symbols, and $\mathcal{L}_{kr^*} := \{ds, ll, rv, \Delta^-_{<_{lex}}, \Delta^-_{\trianglelefteq}, \Delta^-_{len}\}$. We can give $\mathcal{L}_{kr^*}$-structure on $\omega^{<\omega}$ by interpreting $\mathcal{L}_{kr^*}$ as follows.

(i) $\eta <_{ds} \nu$ if $\eta <_{lex} \nu$ and $\{\eta, \nu\}$ forms a direct sibling set.

(ii) $\eta <_{ll} \nu$ if $\eta <_{lex} \nu$ and $\{\eta, \nu\}$ forms a strict left-leaning path.

(iii) $\eta <_{rv} \nu$ if $\eta <_{lex} \nu$ and $\{\eta, \nu\}$ forms a strict right-veering path.

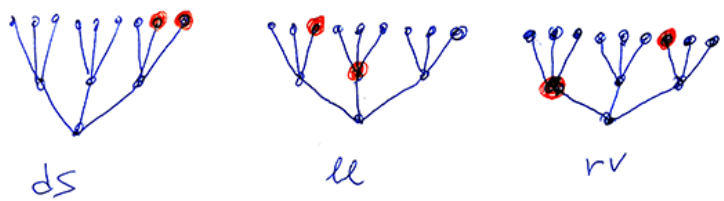
(iv) $(\eta_0, \eta_1, \eta_2, \eta_3) \in \Delta^-_{\trianglelefteq}$ if $\eta_0^- \wedge \eta_1^- \trianglelefteq \eta_2^- \wedge \eta_3^-$.

(v) $(\eta_0, \eta_1, \eta_2, \eta_3) \in \Delta^-_{lex}$ if $\eta_0^- \wedge \eta_1^- <_{lex} \eta_2^- \wedge \eta_3^-$.

(vi) $(\eta_0, \eta_1, \eta_2, \eta_3) \in \Delta^-_{len}$ if $\eta_0^- \wedge \eta_1^- <_{len} \eta_2^- \wedge \eta_3^-$.

where η^- is the $\trianglelefteq$-predecessor of η.

Then the $\mathcal{L}_{kr^*}$-structure on $\omega^{<\omega}$ has modeling property. Let $X := \{0^i : i < \omega\}$ and $Y := \{\langle i \rangle : i < \omega\}$. Then $(\mathcal{L}_{kr^*}, \omega^{<\omega}, X, Y)$ satisfies n-variable theorem and $D_{simple} = D_{X,Y}^{\mathcal{L}_{kr^*}, \omega^{<\omega}}$.



We can expand the $\mathcal{L}_{kr^*}$-structure on $\omega^{<\omega}$ to a randomly coloured $\mathcal{L}_{kr^*} \cup \{C\}$-structure and get $D_{stable} = D_{X,Y}^{\mathcal{L}^C_{kr^*}, \omega^{<\omega}}$.

Higher arity

Fact [Day]

- NIP can be characterized by coloured version of TP_2.
- Stability can be characterized by coloured version of TP.

Question

Is there a tree-property whose coloured version characterizes NIP_k?

Definition

Let $m, k \in \omega$. We say $\varphi(x, y)$ has TP_2 if there exists $(b_{n,m})_{n,m<\omega}$ such that

(i) $\{\varphi(x, b_{n,m_l})\}_{l<2}$ is inconsistent for all $n < \omega$ and $(m_l)_{l<2}$ with $m_0 < m_1$.

(ii) for each $f : \omega \to \omega$,

$$\{\varphi(x, b_{n,f(n)})\}_{n<\omega}$$

is consistent.

Definition

Let $m, k \in \omega$. We say $\varphi(x, y_0, ..., y_{k-1})$ has TP_2^k if there exists $(b^0_{n,m})_{n,m<\omega}, ..., (b^{k-1}_{n,m})_{n,m<\omega}$ such that

(i) $\{\varphi(x, b^0_{n_0,m_{0,l}}, ..., b^{k-1}_{n_{k-1},m_{k-1,l}})\}_{l<2}$ is inconsistent for all $n_0, ..., n_{k-1} < \omega$ and $(m_{0,l})_{l<2}$, ..., $(m_{k-1,l})_{l<2}$ with $m_{i,0} < m_{i,1}$ for all $i < k$.

(ii) for each $f : \omega^k \to \omega^k$,

$$\{\varphi(x, b^0_{n_0,\pi_0\circ f(n_0,...,n_{k-1})}, ..., b^{k-1}_{n_{k-1},\pi_{k-1}\circ f(n_0,...,n_{k-1})})\}_{n_0,...,n_{k-1}<\omega}$$

is consistent, where $\pi_i : \omega^k \to \omega$ is the projection map.

We say T has TP_2^k if φ has it for some φ. We say T is NTP_2^k if it does not have TP_2^k.

Higher arity

Let $\mathcal{L}_{TP_2^k} := \{<, E, P_0, ..., P_{k-1}\}$. Let $\mathcal{I}_{TP_2^k}$ be an $\mathcal{L}_{TP_2^k}$-structure on $k \times \omega \times \omega$ such that

(i) $(i, n, m) < (i', n', m')$ if and only if one of the following holds
- $i < i'$,
- $i = i'$ and $n < n'$,
- $i = i'$, $n = n'$, and $m < m'$,

(ii) $E((i, n, m), (i', n', m'))$ if and only if $i = i'$ and $n = n'$,

(iii) $P_i((j, n, m))$ if and only if $i = j$.

Let

$$X := \{((0, n_0, 0), ..., (k-1, n_{k-1}, 0))\}_{(n_0,...,n_{k-1}) \in \omega^k}$$

and

$$Y := \{\ ((0,0,0), ..., (k-1,0,0)),\ ((0,0,1), ..., (k-1,0,1))\ \}.$$

Then $(\mathcal{L}_{TP_2^k}, \mathcal{I}_{TP_2^k}, k, X, Y)$ satisfies n-variable theorem and $D_{NTP_2^k} = D_{k,X,Y}^{\mathcal{L}_{TP_2^k}, \mathcal{I}_{TP_2^k}}$.

We can expand the $\mathcal{L}_{TP_2^k}$-structure $\mathcal{I}_{TP_2^k}$ to a randomly coloured $\mathcal{L}_{TP_2^k} \cup \{C\}$-structure and get $D_{NIP_k} = D_{k,X,Y}^{\mathcal{L}_{TP_2^k}^C, \omega^{<\omega}}$.

Higher arity

Theorem [K.]

Let $D \in \mathfrak{D}^h_{n\text{-var}}$ and T be a theory. Additionally, we assume that acl $=$ dcl in T. Then $T \in D$ if and only if $T^{gt} \in D$.

Corollary

Suppose that acl $=$ dcl in T. Then T is NIP$_k$ if and only if T^{gt} is NIP$_k$.

Theorem [K.]

Let $D \in \mathfrak{D}^h_{n\text{-var}}$. Then $T \in D$ if and only if $T^{\delta}_g \in D$.

Corollary

T is NIP$_k$ if and only if T^{δ}_g is NIP$_k$.

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