n-variable theorem and its applications to preservation problems

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① We define a class of dividing lines $\mathfrak{D}_{n\text{-var}}$ in terms of positive consistency-inconsistency patterns and generalized indiscernibles. And we will show that many dividing lines belong to this class.

② If a dividing line D belongs to $\mathfrak{D}_{n\text{-var}}$ and a first-order theory T does not belongs to D, then we can always find a witness $\varphi(\bar{x}, \bar{y})$ of $T \notin D$ satisfying *n-variable theorem*:

The type $\{\varphi(x, a_\eta)\}_{\eta \in X}$ which is required to be consistent in the configuration has a realization whose (model-theoretic) algebraic dimension is $|x|$ (i.e., $\dim(\{\varphi(x, a_\eta)\}_{\eta \in X}) = |x|$).

It is a weaker version of one-variable theorem.

③ n-variable theorem is useful when we solve *preservation problems*, a question asking whether a certain dividing line is closed under taking a certain model-theoretic construction.

By a *dividing line*, we mean a class of complete first-order theories.

There are many different ways defining dividing lines.

- Existence of a certain pattern of definable sets (NTP, NTP_1, NTP_2)
- Existence of a certain independence notion (stable, simple, NSOP_1, rosy)
- Collapsing indiscernible (stable, NIP, k-NIP, NFOP_2)
- Using rank notions (dp-minimal, SU-rank)
- Restricting shapes of definable sets (o-minimal, C-minimal)

Preservation problem

Given a dividing line D and a model-theoretic construction $T \mapsto T'$, does the construction preserve membership in D? Is the following implication true?

$$T \in D \;\Rightarrow\; T' \in D.$$

Shelah expansions

$$\mathrm{Th}(M) \text{ is NIP} \quad \Rightarrow \quad \mathrm{Th}(M^{\mathrm{Sh}}) \text{ is NIP.}$$

We can name all externally definable sets and continue to use NIP tools in the expanded language.

Generic automorphisms

$$T \text{ is superstable} \quad \Rightarrow \quad T_A \text{ is supersimple.}$$

We can add a generic automorphism and use ordinal-valued SU-rank and induction on rank in the expanded theory.

Pfaffian closure

Let $\mathcal{R}$ be an expansion of the real field.

$$\mathcal{R} \text{ is o-minimal} \quad \Rightarrow \quad \mathcal{P}(\mathcal{R}) \text{ is o-minimal,}$$

where $\mathcal{P}(\mathcal{R})$ is the Pfaffian closure of $\mathcal{R}$. We can add predicates for all Rolle leaves of definable C^1 differential 1-forms while retaining the geometric properties of o-minimal structures, such as cell decomposition and dimension theory.

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Let I be a set and $\mathfrak{X}, \mathfrak{Y} \subseteq \mathcal{P}(I) \times \mathcal{P}(I)$. Let D be the class of first-order theories T such that there exist $\varphi(\bar{x}, \bar{y})$ and parameters $(a_\eta)_{\eta \in I}$ such that

$$\{\varphi(\bar{x}, a_\eta) : \eta \in X^+\} \cup \{\neg\varphi(\bar{x}, a_\eta) : \eta \in X^-\}$$

is consistent for all $(X^+, X^-) \in \mathfrak{X}$ and

$$\{\varphi(\bar{x}, a_\eta) : \eta \in Y^+\} \cup \{\neg\varphi(\bar{x}, a_\eta) : \eta \in Y^-\}$$

is inconsistent for all $(Y^+, Y^-) \in \mathfrak{Y}$.

A dividing line D defined in this way is said to be *straightly definable*.

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Example

Let $I = \omega$, $\mathfrak{X} = \{(\{j \in \omega : i < j\}, \{j \in \omega : j \leq i\}) : i \in \omega\}$, and $\mathfrak{Y} = \emptyset$. Then

$$T \in D \quad \Leftrightarrow \quad T \text{ is unstable.}$$

Example

Let $I = \omega$, $\mathfrak{X} = \{(A, \omega \setminus A) : A \subseteq \omega\}$, and $\mathfrak{Y} = \emptyset$. Then

$$T \in D \quad \Leftrightarrow \quad T \text{ has IP.}$$

Let I be a set and $\mathfrak{X}, \mathfrak{Y} \subseteq \mathcal{P}(I) \times \mathcal{P}(I)$. Let D be the class of first-order theories T such that there exist $\varphi(\bar{x}, \bar{y})$ and parameters $(a_\eta)_{\eta \in I}$ such that

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is inconsistent for all $(Y^+, Y^-) \in \mathfrak{Y}$.

A dividing line D defined in this way is said to be *straightly definable*.

If $X^- = \emptyset$ and $Y^- = \emptyset$ for all $(X^+, X^-) \in \mathfrak{X}$ and $(Y^+, Y^-) \in \mathfrak{Y}$, then we say that the straight pattern is *positive*.

Example

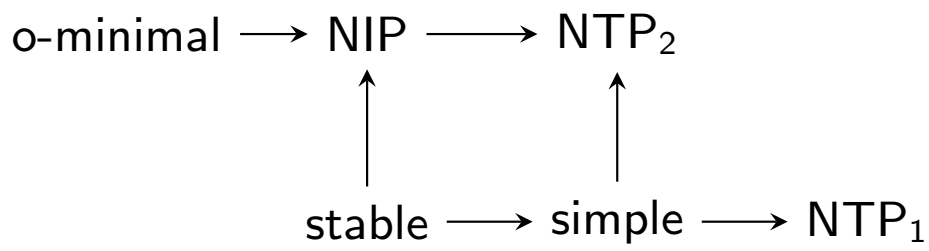
Let $I = \omega^{<\omega}$, $\mathfrak{X} = \{(X, \emptyset) : X \text{ is a path}\}$, and $\mathfrak{Y} = \{(Y, \emptyset) : Y \text{ is a set of two incomparable nodes}\}$. Then

$$T \in D \quad \Leftrightarrow \quad T \text{ has } \mathrm{TP}_1.$$

Let T be a complete theory and M its model.

- [Shelah] We say T is *stable* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_i)_{i<\omega}$, and $(b_i)_{i<\omega}$ such that
$$\models \varphi(a_i, b_j) \text{ if and only if } i < j.$$
- [Shelah] We say T has *NIP* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_i)_{i<\omega}$, and $(b_I)_{I\subseteq\omega}$ such that
$$\models \varphi(a_i, b_I) \text{ if and only if } i \in I.$$
- [Shelah] We say T is *simple* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta\in X}$ is consistent if X is a path in $\omega^{<\omega}$.
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta\in Y}$ is consistent if Y is a set of direct siblings.
- [Shelah] We say T is *NTP*$_1$ if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta\in X}$ is consistent if and only if X is a path in $\omega^{<\omega}$.
- [Shelah] We say T is *NTP*$_2$ if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_{i,j})_{i,j<\omega}$ such that
  $\{\varphi(\bar{x}, a_{i,f(i)})\}_{j<\omega}$ is consistent for all $f : \omega \to \omega$.
  $\{\varphi(\bar{x}, a_{i,j})\}_{j<\omega}$ is inconsistent for all $i < \omega$.

The following diagram is well known.

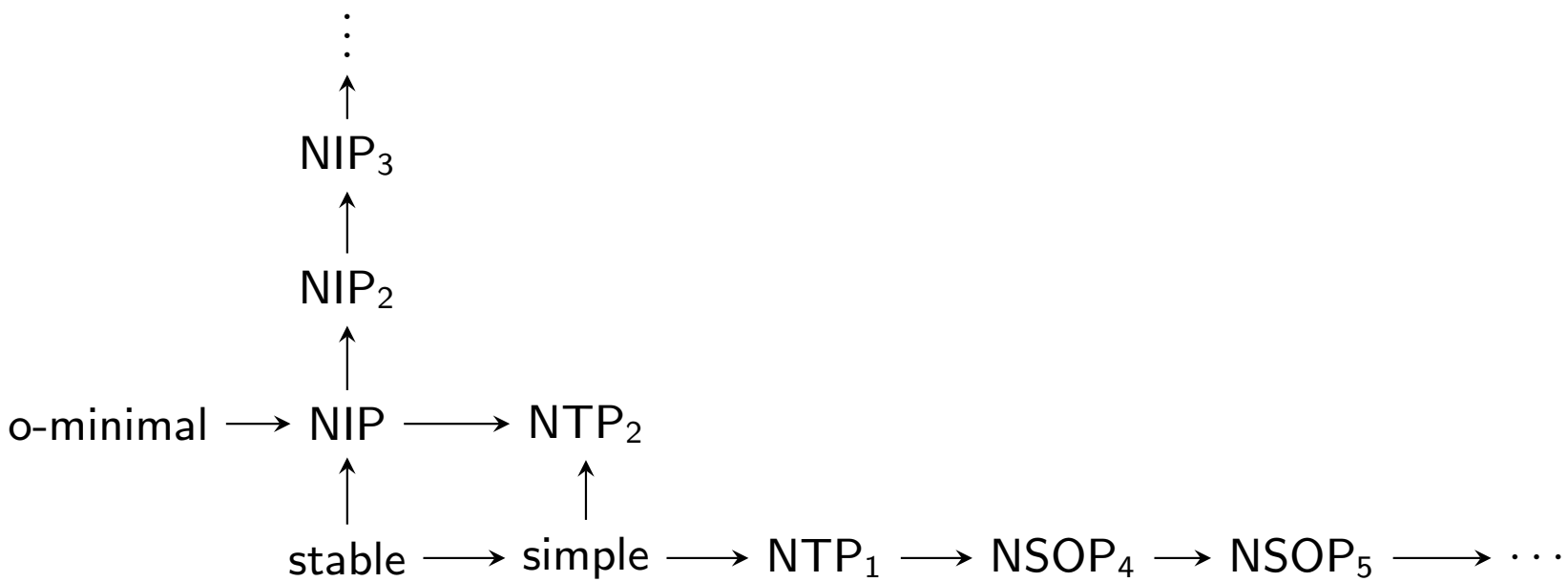


This diagram means that:

- if a theory is stable, then it is NIP and simple,
- if a theory is simple, then it is NTP$_1$ and NTP$_2$,
- if a theory is o-minimal, then it is NIP,
- if a theory is NIP, then it is NTP$_2$.

Let $2 < n < \omega$ and $1 < k < \omega$.

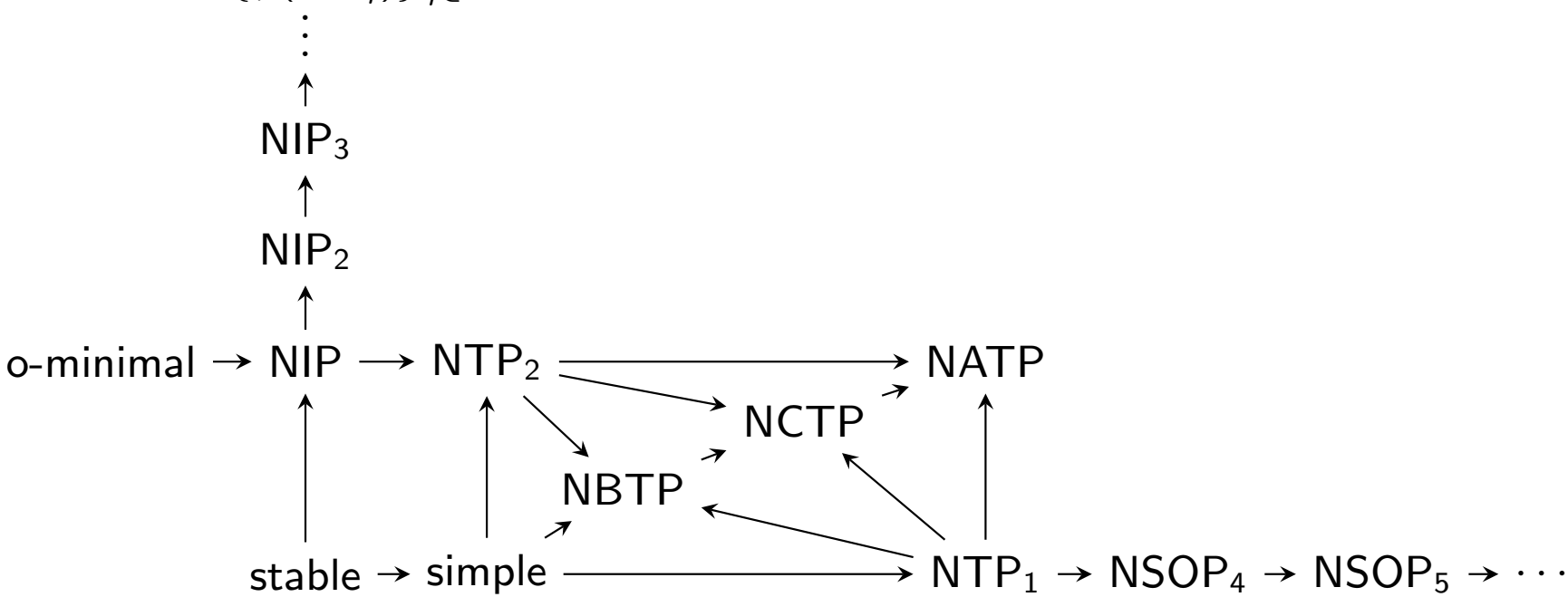
- [Shelah] We say T is $NSOP_n$ if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, a_{i+1})$ for all $i < \omega$ and $\{\varphi(\bar{x}_0, \bar{x}_1), ..., \varphi(\bar{x}_{n-2}, \bar{x}_{n-1}), \varphi(\bar{x}_{n-1}, \bar{x}_0)\}$ is inconsistent.
- [Shelah] We say T has NIP_k if there do not exist $\varphi(\bar{x}, \bar{y}_0, ..., \bar{y}_{k-1})$, $(a_i^0, ..., a_i^{k-1})_{i<\omega}$, and $(b_I)_{I \subseteq \omega^k}$ such that $\models \varphi(b_I, a_{i_0}^0, ..., a_{i_{k-1}}^{k-1})$ if and only if $(i_0, ..., i_{k-1}) \in I$.



- [Ahn, K.] We say T is *NATP* if there are no $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is inconsistent if and only if X is a path in $\omega^{<\omega}$.

- [Mutchnik] We say T is *NCTP* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent if X is a descending comb.
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in Y}$ is inconsistent if Y is a path.

- [Ramsey, Kruckman] We say T is *NBTP* if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent if X is a left-leaning path.
  $\{\varphi(\bar{x}, a_\eta)\}_{\eta \in Y}$ is inconsistent if Y is a right-veering path.

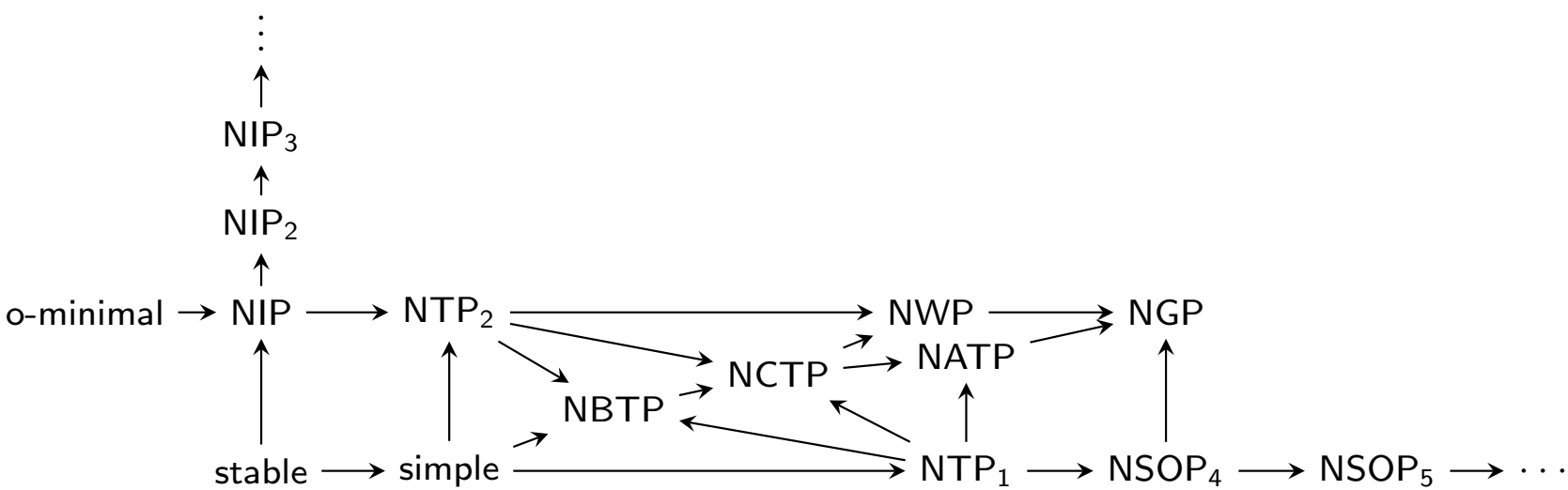


- [Hanson] For $k < \omega$, we say T is *NWP (having no $(k,1,1)$-weave of depth ω)* if there do not exist $\varphi(\bar{x},\bar{y})$ and $(a_\eta)_{\eta\in(2^2)^\omega}$ such that

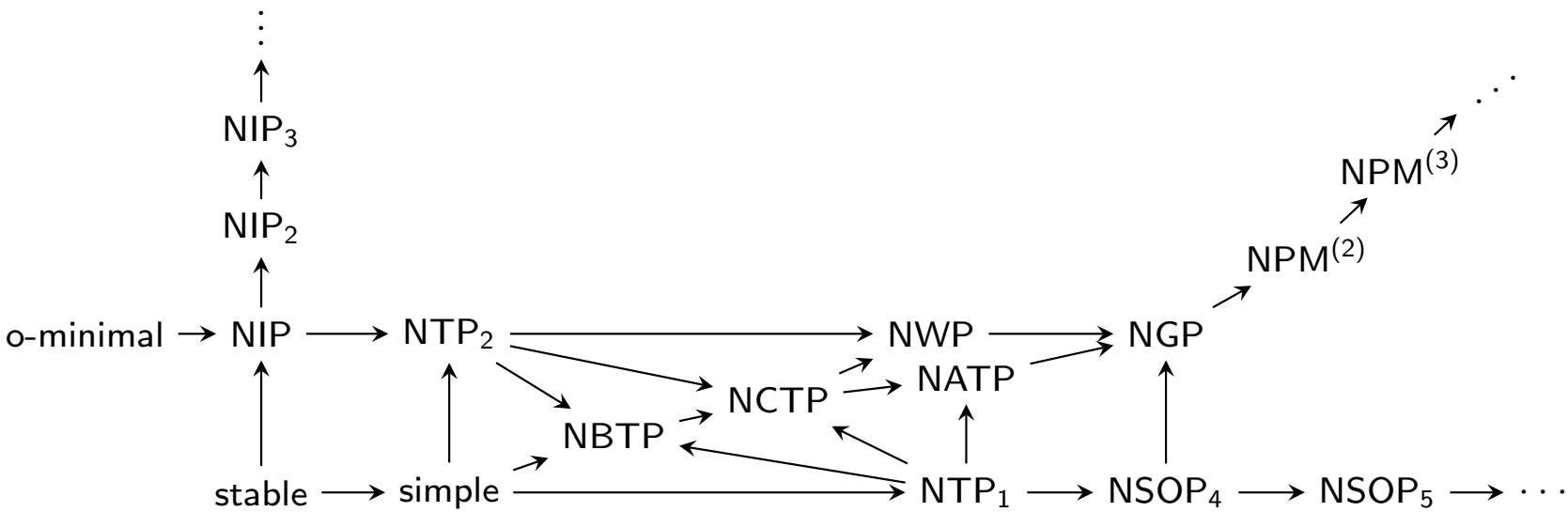
  $\{\varphi(\bar{x},a_\eta)\}_{\eta\in X}$ is consistent for each finite right-1-comb X,
  $\{\varphi(\bar{x},a_\eta)\}_{\eta\in X}$ is k-inconsistent for each finite up-1-comb X.

- [Hanson] For $k < \omega$, we say T has an *NGP (having no infinite k-grid)* if there do not exist $\varphi(\bar{x},\bar{y})$ and $(a_{i,j})_{i,j<\omega}$ such that

  * $\{\varphi(\bar{x},a_{i,j})\}_{(i,j)\in X}$ is consistent if X satisfies $(i,j),(i',j')\in X \Rightarrow i\le i'\wedge j\le j'$ or $i'\le i\wedge j'\le j$,
  * $\{\varphi(\bar{x},a_{i,j})\}_{(i,j)\in X}$ is k-inconsistent if X satisfies $(i,j),(i',j')\in X \Rightarrow i\not\le i'\vee j\not\le j'$ and $i'\not\le i\vee j'\not\le j$.



- [Bailetti] For $1 < k < \omega$, we say T is *$NPM^{(k)}$ (having no $PM^{(k)}$)* if there do not exist $\varphi(\bar{x}, \bar{y})$ such that for a generic k-hypergraph (ω, E) on ω, there exists $(b_i)_{i<\omega}$ such that
  - $\{\varphi(\bar{x}, b_i)\}_{i \in X}$ is consistent if $X \subseteq \omega$ is a clique (i.e., $E(i_0, ..., i_{k-1})$ for all $i_0, ..., i_{k-1} \in X$),
  - $\{\varphi(\bar{x}, b_i)\}_{i \in X}$ is inconsistent if $X \subseteq \omega$ contains $i_0, ..., i_{k-1}$ such that $\neg E(i_0, ..., i_{k-1})$.



① We will see that all of these dividing lines in the diagram, except for o-minimality and NSOP_n for $n > 3$, can be formulated in a uniform way.

Definition

Let $\mathcal{L}^*$ be a language and $\mathcal{I}^*$ an $\mathcal{L}^*$-structure. Let $\mathcal{L}$ be another language, T an $\mathcal{L}$-theory, $\mathbb{M}$ a monster model of T, and A a small subset of $\mathbb{M}$.
A family $(a_\eta)_{\eta \in \mathcal{I}^*}$ in $\mathbb{M}$ is said to be *$\mathcal{I}^*$-indiscernible over A* if for all finite tuples $\bar{\eta}, \bar{\nu}$ from $\mathcal{I}^*$,

$$\mathrm{qftp}_{\mathcal{L}^*}(\bar{\eta}) = \mathrm{qftp}_{\mathcal{L}^*}(\bar{\nu}) \quad \Rightarrow \quad \mathrm{tp}_{\mathcal{L}}(a_{\bar{\eta}}) = \mathrm{tp}_{\mathcal{L}}(a_{\bar{\nu}}).$$

Example

Ordinary indiscernible sequences correspond to the case where $\mathcal{L}^* = \{<\}$ and $\mathcal{I}^*$ is a linear order.

$$i_0 < \cdots < i_n \text{ and } j_0 < \cdots < j_n \;\Rightarrow\; a_{i_0} \ldots a_{i_n} \equiv a_{j_0} \ldots a_{j_n}$$

Roughly speaking, a generalized indiscernible sequence is an $\mathcal{I}^*$-indexed set of parameters $(a_\eta)_{\eta \in \mathcal{I}^*} \subseteq \mathbb{M}$. If two subtuples of $(a_\eta)_{\eta \in \mathcal{I}^*}$ are given, then they have the same *$\mathcal{L}$-type* in $\mathbb{M}$ when their index have the same *quantifier-free $\mathcal{L}^*$-type* in $\mathcal{I}^*$.

Definition

Let $\mathcal{L}$ be an arbitrary language, T an arbitrary complete $\mathcal{L}$-theory, and $M \models T$ a sufficiently saturated model. Let $\mathcal{L}^*$ be a language and $\mathcal{I}^*$ an $\mathcal{L}^*$-structure. We say an $\mathcal{L}^*$-structure $\mathcal{I}^*$ has *modeling property* if for any $(a_\eta)_{\eta \in \mathcal{I}^*} \subseteq M$, there exists $(b_\eta)_{\eta \in \mathcal{I}^*}$ such that

- $(b_\eta)_{\eta \in \mathcal{I}^*}$ is $\mathcal{I}^*$-indiscernible,
- for any $\bar{\eta} \in \mathcal{I}^*$ and $\varphi(\bar{x}) \in \mathcal{L}$, there exists $\bar{\nu}$ such that $\mathrm{qftp}_{\mathcal{L}^*}(\bar{\eta}) = \mathrm{qftp}_{\mathcal{L}^*}(\bar{\nu})$ and $M \models \varphi(b_{\bar{\nu}}) \leftrightarrow \varphi(a_{\bar{\nu}})$.

In this case, we say $(b_\eta)_{\eta \in \mathcal{I}^*}$ is *locally based* on $(a_\eta)_{\eta \in \mathcal{I}^*}$.

Example

Let $\mathcal{L}^* := \{<\}$ and $\mathcal{I}^*$ be an infinite linear order. Let $\mathcal{L}$ be another language and $\mathbb{M}$ an $\mathcal{L}$-structure. If $(a_i)_{i \in \mathcal{I}^*}$ is a sequence of elements in $\mathbb{M}$, then there exists $(b_i)_{i \in \mathcal{I}^*} \subseteq \mathbb{M}$ such that

- For $\varphi(x_0, \ldots x_n) \in \mathcal{L}$ and $i_0 < \cdots < i_n$ in $\mathcal{I}^*$, there exist $j_0 < \cdots < j_n$ such that $\models \varphi(a_{i_0}, \ldots, a_{i_n}) \leftrightarrow \varphi(b_{j_0}, \ldots, b_{j_n})$.
- $(b_i)_{i \in \mathcal{I}^*}$ is $\mathcal{I}^*$-indiscernible (usual order-indiscernible).

Let ω^n be the set of maps from n to ω and $\omega^{<\omega} := \bigcup_{n<\omega} \omega^n$.
We can define binary relations $\lhd, <_{lex}$ and a binary funtion $\wedge$ on $\omega^{<\omega}$.

- $\eta \lhd \nu$ if $\eta \subseteq \nu$.
- $\eta <_{lex} \nu$ if (i) $\eta \lhd \nu$ or
  (ii) $\xi^\frown(i) \trianglelefteq \eta$ and $\xi^\frown(j) \trianglelefteq \nu$ for some $\xi \in \omega^{<\omega}$ and $i < j < \omega$.
- $\eta \wedge \nu = \xi$ if $\xi \trianglelefteq \eta$, $\xi \trianglelefteq \nu$, and ξ is the $\lhd$-largest such element.

Let $\mathcal{L}_0 := \{\lhd, <_{lex}, \wedge\}$, and consider the $\mathcal{L}_0$-structure on $\omega^{<\omega}$ described above.

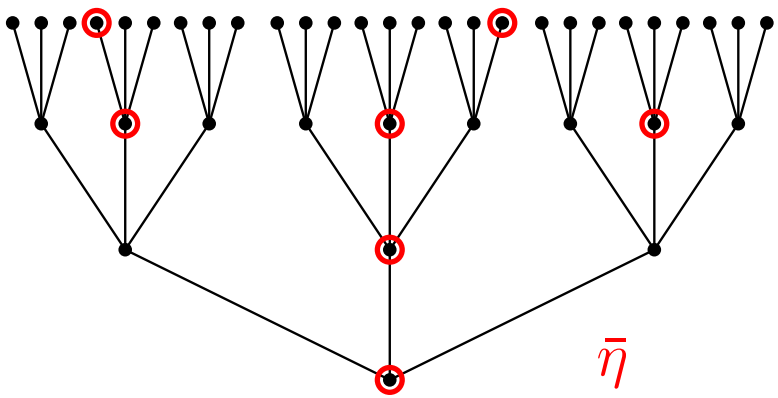
Fact [Takeuchi, Tsuboi] & [Kim, Kim, Scow]

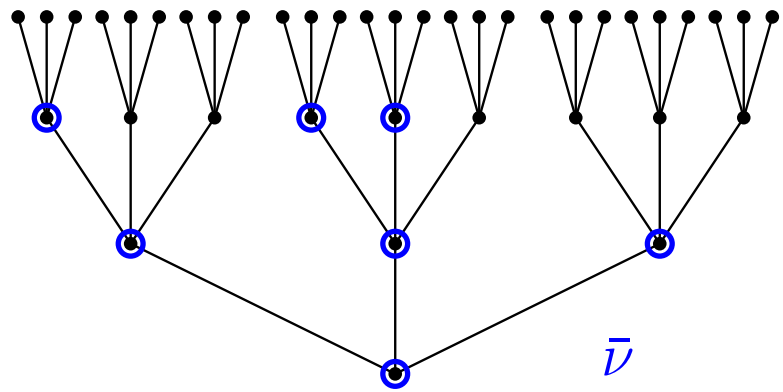
$\omega^{<\omega}$ has modeling property.

If $(a_\eta)_{\eta\in\omega^{<\omega}} \subseteq \mathbb{M}$ is given, then there exists $\omega^{<\omega}$-indiscernible $(b_\eta)_{\eta\in\omega^{<\omega}}$ such that

for any $\varphi(\bar{x}) \in \mathcal{L}$ and $\bar{\eta} \in \omega^{<\omega}$, there exists $\bar{\nu}$ such that

$$\bar{\eta} \sim_{\mathcal{L}_0} \bar{\nu} \quad \text{and} \quad \models \varphi(a_{\bar{\eta}}) \leftrightarrow \varphi(b_{\bar{\nu}}).$$





Let I be a set and $\mathfrak{X}, \mathfrak{Y} \subseteq \mathcal{P}(I) \times \mathcal{P}(I)$. Let D be the class of first-order theories T such that there exist $\varphi(\bar{x}, \bar{y})$ and parameters $(a_\eta)_{\eta \in I}$ such that

$$\{\varphi(\bar{x}, a_\eta) : \eta \in X^+\} \cup \{\neg\varphi(\bar{x}, a_\eta) : \eta \in X^-\}$$

is consistent for all $(X^+, X^-) \in \mathfrak{X}$ and

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is inconsistent for all $(Y^+, Y^-) \in \mathfrak{Y}$.

A dividing line D defined in this way is said to be *straightly definable*.

If $X^- = \emptyset$ and $Y^- = \emptyset$ for all $(X^+, X^-) \in \mathfrak{X}$ and $(Y^+, Y^-) \in \mathfrak{Y}$, then we say that the straight pattern is *positive*.

Positive straight definitions consist of three part.

- The index set I.
- A set $\mathfrak{X}$ of subsets of I.
- A set $\mathfrak{Y}$ of subsets of I.

Once we choose them, then we can ask if a given theory T has a formula $\varphi(x, y)$ and parameters $(a_\eta)_{\eta \in I}$ such that $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for each $X \in \mathfrak{X}$ and $\{\varphi(x, a_\eta)\}_{\eta \in Y}$ for each $Y \in \mathfrak{Y}$. The existence of such φ and $(a_\eta)_{\eta \in I}$ characterizes a class of first-order theories.

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In this point of view, some dividing lines are characterized by a triple $(I, \mathfrak{X}, \mathfrak{Y})$ (e.g., TP_1 is characterized by $(\omega^{<\omega},$ *the set of paths*, *the set of incomparable pairs*$)$).

We want to add one more ingredient in this picture, namely a language $\mathcal{L}^*$, which allows us to view I as a structure, not merely a set.

From now, we will consider quadruples $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ such that

- $\mathcal{L}^*$ is a language,
- $\mathcal{I}^*$ is an $\mathcal{L}^*$-structure,
- X is a subset of $\mathcal{I}^*$ $(|\mathfrak{X}| = 1)$,
- $\bar{Y}$ is a finite set of subsets of $\mathcal{I}^*$ $(|\mathfrak{Y}| < \omega)$.

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- $\mathcal{L}^*$ is a language,
- $\mathcal{I}^*$ is an $\mathcal{L}^*$-structure,
- X is a subset of $\mathcal{I}^*$,
- $\bar{Y}$ is a finite set of subsets of $\mathcal{I}^*$.

This quadruple characterizes a dividing line in essentially the same way as a positive straight definition. We want to go further and understand which good properties of the resulting dividing line follow from the structural properties of $\mathcal{I}^*$.

Definition

We say $\mathcal{I}^*$ is *weakly monochromatic* if $\{\langle\eta\rangle : \eta \in \mathcal{I}^*\}$ has only finitely many elements up to $\mathcal{L}^*$-isomorphism.

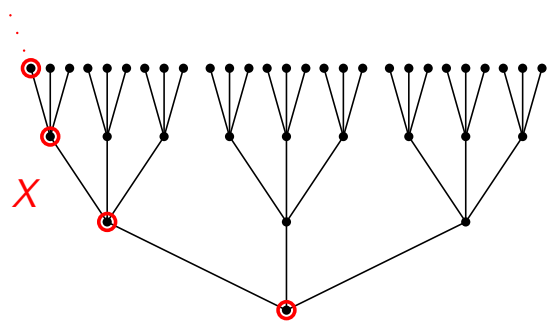
Definition

We say a subset X of $\mathcal{I}^*$ is *cofinal* in $\mathcal{I}^*$ if for any finite subset Z of $\mathcal{I}^*$, there exists an $\mathcal{L}^*$-embedding $f : \mathcal{I}^* \to \mathcal{I}^*$ such that

- $f(\mathcal{I}^*) \cap Z = \emptyset$,
- $\bar{\eta} \sim_{\mathcal{L}^*} \bar{\nu} \Rightarrow \bar{\eta}Z \sim_{\mathcal{L}^*} \bar{\nu}Z$ for all $\bar{\eta}, \bar{\nu} \in f(\mathcal{I}^*)$ with $|\bar{\eta}| = |\bar{\nu}| < \omega$,
- $\mathrm{age}_{\mathcal{L}^*}(f(\mathcal{I}^*) \cap X) = \mathrm{age}_{\mathcal{L}^*}(X)$.

If X is cofinal, then, roughly speaking, X is sufficiently spread so that finitely many obstacles do not harm the indiscernibility (by considering an isomorphic image of the whole structure).

Let $X := \{(), (0), (00), (000), (0000), ...\}$. Then for any finitely many $\eta_0, ..., \eta_n$, we can find a sufficiently large natural number m such that $f : \nu \mapsto (0^m)^\frown\nu$ satisfies all conditions above. Thus X is cofinal in $\omega^{<\omega}$.



Definition

Let $J \subseteq \mathcal{I}^*$. We say $\mathcal{I}^*$ is *monochromatically extendable over J* if for every cardinal λ, there exists an $\mathcal{L}^*$-structure $\mathcal{I}^\dagger$ such that

- $\mathrm{age}_{\mathcal{L}^*}(\mathcal{I}^\dagger) = \mathrm{age}_{\mathcal{L}^*}(\mathcal{I}^*)$,
- for every coloring $c : \mathcal{I}^\dagger \to \lambda$, there exists an $\mathcal{L}^*$-embedding $f : \mathcal{I}^* \to \mathcal{I}^\dagger$ such that

$$\langle \eta \rangle \sim_{\mathcal{L}^*} \langle \nu \rangle \quad \Longrightarrow \quad c(f(\eta)) = c(f(\nu)) \qquad (\eta, \nu \in J).$$

When $J = \mathcal{I}^*$, we simply say that $\mathcal{I}^*$ is *monochromatically extendable*.

Example

The $\{\wedge, <_{lex}, \lhd\}$-structure on $\omega^{<\omega}$ is monochromatically extendable. For any cardinal λ, we can find a sufficiently large cardinal $\kappa > \lambda$ such that for any coloring $\omega^{<\kappa} \to \lambda$, there exists an $\{\wedge, <_{lex}, \lhd\}$-embedding $f : \omega^{<\omega} \to \omega^{<\kappa}$ such that $c(f(\eta)) = c(f(\nu))$ for all $\eta, \nu \in \omega^{<\omega}$.

Definition

We say a quadruple $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfies the *n-variable theorem* if

(i) $\mathcal{L}^*$ is a language, $\mathcal{I}^*$ is a locally finite $\mathcal{L}^*$-structure having the modeling property, X is a subset of $\mathcal{I}^*$, and $\bar{Y}$ is a finite set of subsets of $\mathcal{I}^*$,

(ii) $\mathcal{I}^*$ is weakly monochromatic,

(iii) X is cofinal in $\mathcal{I}^*$,

(iv) $\mathcal{I}^*$ is monochromatically extendable over

$$\mathbb{X} := \{\eta \in \mathcal{I}^* : \langle \eta \rangle \sim_{\mathcal{L}^*} \langle \nu \rangle \text{ for some } \nu \in X\},$$

(v) $Y \subseteq \mathbb{X}$ for each $Y \in \bar{Y}$,

(vi) $|Y| > 1$ for each $Y \in \bar{Y}$,

(vii) there exists a disjoint union $W \subseteq \mathcal{I}^*$ of X' and Y' such that

- $X' \sim^{\text{fin}}_{\mathcal{L}^*} X$ and $Y' \sim^{\text{fin}}_{\mathcal{L}^*} Y$ for some $Y \in \bar{Y}$,
- $\{\eta\} \cup X' \sim^{\text{fin}}_{\mathcal{L}^*} X$ for each $\eta \in Y'$.

If $\bar{Y} = \{Y\}$, then we just say $(\mathcal{L}^*, \mathcal{I}^*, X, Y)$ satisfies the n-variable theorem.

Definition

For a given quadruple $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfying n-variable theorem, let $D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}}$ be the class of all first-order theories T such that there do not exist $\varphi(x, y)$ in the language of T and $(a_\eta)_{\eta\in\mathcal{I}^*}$ in a model of T such that

(i) $(a_\eta)_{\eta\in\mathcal{I}^*}$ is $\mathcal{I}^*_{\mathcal{L}^*}$-indiscernible,

(ii) $\{\varphi(x, a_\eta)\}_{\eta\in X}$ is consistent,

(iii) $\{\varphi(x, a_\eta)\}_{\eta\in Y}$ is inconsistent for each $Y \in \bar{Y}$.

Let $\mathfrak{D}_{n\text{-var}}$ be a class of classes of first-order theories such that

$$\mathfrak{D}_{n\text{-var}} := \left\{ D^{\mathcal{L}^*,\mathcal{I}^*}_{X,\bar{Y}} : (\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y}) \text{ satisfies } n\text{-variable theorem} \right\}.$$

Let D_{stable} be the class of stable theories. Similarly we define $D_{o-minimal}, D_{simple}, D_{NIP}, D_{NTP_1}, D_{NTP_2}, \ldots$.

Observation

$D_{stable}, D_{simple}, D_{NIP}, D_{NTP_1}, D_{NTP_2}, D_{NATP}, D_{NCTP}, D_{NBTP}, D_{NGP}, D_{NWP}, D_{NPM^{(k)}} \in \mathfrak{D}_{n\text{-var}}$

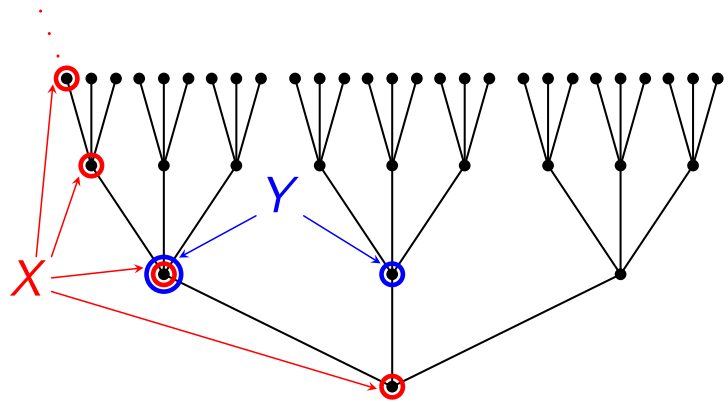
Definition [Shelah]

We say T is NTP_1 if there do not exist $\varphi(\bar{x}, \bar{y})$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

$\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}$ is consistent if and only if X is a path in $\omega^{<\omega}$

(i.e. $X := \{\eta_0, ...\}$ such that $\eta_i \lhd \eta_{i+1}$ for all i).

Choose any uncountable cardinal κ with $\mathrm{cf}(\kappa) = \kappa$ and let $X := \{(), (0), (00), ...\}$ and $Y := \{(0), (1)\}$ in $\omega^{<\kappa}$. Then $(\mathcal{L}_0, \omega^{<\kappa}, X, Y)$ satisfies n-variable theorem and $D_{X,Y}^{\mathcal{L}_0, \omega^{<\kappa}} = D_{NTP_1}$. Thus $D_{NTP_1} \in \mathfrak{D}_{n\text{-var}}$.



Similarly, given a dividing line, we can try to find an appropriate structure with Ramsey property, together with a cofinal subset X and a suitable subset Y witnessing the required inconsistency condition.
This is not always possible. $D_{o-minimal} \notin \mathfrak{D}_{n\text{-var}}$.

Definition

Suppose $T \notin D^{\mathcal{L};\mathcal{I}^*}_{X,\bar{Y}}$. Then there exist $\varphi(\bar{x},\bar{y})$ and $\mathcal{I}^*$-indiscernible $(a_\eta)_{\eta\in\mathcal{I}^*}$ such that

(i) $\{\varphi(\bar{x},a_\eta)\}_{\eta\in X}$ is consistent,

(ii) $\{\varphi(\bar{x},a_\eta)\}_{\eta\in Y}$ is inconsistent for each $Y \in \bar{Y}$.

where $\bar{x}$ and $\bar{y}$ are tuples of variables ($\bar{x}$ is of the form $(x_0,\ldots,x_n)$).
We say φ *witnesses* $T \notin D^{\mathcal{L};\mathcal{I}^*}_{X,\bar{Y}}$ with $(a_\eta)_{\eta\in\mathcal{I}^*}$).

Question

Suppose $T \notin D^{\mathcal{L};\mathcal{I}^*}_{X,\bar{Y}}$. Is there a witness $\varphi(\bar{x},\bar{y})$ of $T \notin D^{\mathcal{L};\mathcal{I}^*}_{X,\bar{Y}}$ such that $|\bar{x}| = 1$?

One-variable theorem

- [Chernikov, Ramsey] If T has TP_1, then there exists $\varphi(\bar{x}, \bar{y})$ witnessing TP_1 such that $|\bar{x}| = 1$.
- [Chernikov] If T has TP_2, then there exists $\varphi(\bar{x}, \bar{y})$ witnessing TP_2 such that $|\bar{x}| = 1$.
- [Ahn, K., Lee] If T has NATP, then there exists $\varphi(\bar{x}, \bar{y})$ witnessing NATP such that $|\bar{x}| = 1$.

Definition (algebraic dimension)

By $\dim(\bar{b}/A)$, we mean
$\max\{|\bar{b}^0 = (b^0_0, ..., b^0_{n-1})| : \bar{b}^0 \subseteq \bar{b}$ and $b^0_i \notin \mathrm{acl}(\bar{b}^0 \setminus \{b^0_i\} \cup A)$ for each $i < n\}$.

If $\Gamma(\bar{x})$ is a partial type over a set of parameters A, we define the algebraic dimension of the corresponding type-definable set by

$$\dim(\Gamma(\bar{x})) := \max\{\dim(\bar{b}/A) : \bar{b} \models \Gamma(\bar{x})\}.$$

n-variable theorem [K.]

Suppose that $(\mathcal{L}^*, \mathcal{I}^*, X, \bar{Y})$ satisfies n-variable theorem. If $T \notin D^{\mathcal{L}^*, \mathcal{I}^*}_{X,\bar{Y}}$, then there exists $\varphi(\bar{x}, \bar{y})$ and $\mathcal{L}^*$-indiscernible $(a_\eta)_{\eta \in \mathcal{I}^*}$ such that

- φ and $(a_\eta)_{\eta \in \mathcal{I}^*}$ witness $T \notin D^{\mathcal{L}^*, \mathcal{I}^*}_{X,\bar{Y}}$,
- $\dim(\{\varphi(\bar{x}, a_\eta)\}_{\eta \in X}) = |\bar{x}|$.

Remark

If $\varphi(x, y)$ witnesses $\neg D$ with $(a_\eta)_{\eta \in \mathcal{I}^*}$ with $|x| = 1$, then $\{\varphi(x, a_\eta)\}_{\eta \in X}$ has infinitely many solutions, which implies $\dim(\{\varphi(x, a_\eta)\}_{\eta \in X}) = 1 = |x|$.

So "having full dimension ($\dim(\{\varphi(x, a_\eta)\}_{\eta \in X}) = |x|$)" is one of the consequences of $|\bar{x}| = 1$.

In this sense, we can say the n-variable theorem is a weaker version of 1-variable theorem.

Remark

The full-dimension conclusion of the n-variable theorem interacts well with features of geometric theories such as *density properties*, *the exchange property*, *H-independence*, and *relative quantifier elimination*, providing a useful tool for proving preservation results.

Even if a theory is not geometric, we may still be able to apply the n-variable theorem if it has certain good algebraic properties. We will see an example of this later when we outline the preservation result for ACF_T.

Application of n-variable theorem

Let $\mathcal{L}$ be a language. We say an $\mathcal{L}$-theory T is *geometric* if

- it eliminates $\exists^\infty$
  (i.e., for all $\varphi(x, y)$, there exists $\psi_\varphi(y)$ such that $\varphi(x, b)$ is algebraic if and only if $\models \psi_\varphi(b)$),
- acl_T satisfies exchange property.
  (i.e., $a \in \mathrm{acl}_T(bC) \setminus \mathrm{acl}_T(C) \Rightarrow b \in \mathrm{acl}_T(Ca)$)

Application of n-variable theorem

Fix a geometric theory T and a sufficiently saturated model $M \models T$.

Fact [Berenstein, Vassiliev]

There exists $H \subseteq M$ such that

- $h \notin \mathrm{acl}_T(H \setminus \{h\})$ for each $h \in H$,
- H is *dense* (i.e., every non-algebraic type over $A \subseteq M$ has a solution in H),
- H is *codense* (i.e., every non-algebraic type over $A \subseteq M$ has a solution outside $\mathrm{acl}_T(HA)$).

H is said to be independent and dense/codense.

Let P be a new unary predicate symbol, and let $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. Let $H \subseteq M$ be a independent dense/codense subset. Then the $\mathcal{L}_P$-expansion (M, H) of M is called an *H-structure* of M.

Fact [Berenstein, Vassiliev]

The $\mathcal{L}_P$-theory of H-structure is complete. We denote it by T^{ind}.

Application of n-variable theorem

Let $\mathcal{L}$ be a language, T a geometric $\mathcal{L}$-theory, and M a model of T. Let P be a new predicate symbol and $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. We say an $\mathcal{L}_P$-expansion (M, H) of M is a *lovely pair* if H is dense/codense and $H \prec_{\mathcal{L}} M$.

Fact [Berenstein, Vassiliev]

If $H \subseteq M$ is independent dense/codense, then $\mathrm{acl}(H)$ is still dense/codense and $\mathrm{acl}(H) \prec_{\mathcal{L}} M$. The $\mathcal{L}_P$-theory of lovely pair is complete. We denote it by T_P.

For each $\mathcal{L}$-formula $\varphi(\bar{x})$, let $P_\varphi(\bar{x})$ be a new predicate and $\mathcal{L}' := \{P_\varphi : \varphi \in \mathcal{L}\}$. Let (M, H) be an H-structure of M. Then H canonically inherits an $\mathcal{L}'$-structure. We call the $\mathcal{L}'$-theory of this structure the *generic trivialization* of M and denote it by T^{gt}.

T^{gt} can be regarded as a Morleyization on an independence dense/codense subset.

$\mathrm{acl}_{T^{gt}}$ is trivial (i.e., $\mathrm{acl}_{T^{gt}}(x) = x$ for all $x \in M$).

Application of n-variable theorem

Fact [Berenstein, Vassiliev]

- T is stable if and only if T^{gt} is stable.
- [+Kaplan] T is simple if and only if T^{gt} is simple.
- T is NIP if and only if T^{gt} is NIP.
- T is NTP_2 if and only if T^{gt} is NTP_2.

Fact [Berenstein, Vassiliev]

- T is stable if and only if T^{ind} is stable if and only if T_P is stable.
- T is simple if and only if T^{ind} is simple if and only if T_P is simple.
- T is NIP if and only if T^{ind} is NIP if and only if T_P is NIP.

Fact [Berenstein, Kim]

T is NTP_2 if and only if T^{ind} is NTP_2 if and only if T_P is NTP_2.

Fact [Dobrowolski, Kim]

T is NTP_1 if and only if T^{ind} is NTP_1 if and only if T_P is NTP_1.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$ and T be a theory. Then the following are equivalent.

- $T \in D$.
- $T^{gt} \in D$.

Corollary

- T is NTP_1 if and only if T^{gt} is NTP_1.
- T is NATP if and only if T^{gt} is NATP.
- T is NCTP if and only if T^{gt} is NCTP.
- T is NBTP if and only if T^{gt} is NBTP.
- T is NGP if and only if T^{gt} is NGP .
- T is NWP if and only if T^{gt} is NWP .
- T is $\mathrm{NPM}^{(k)}$ if and only if T^{gt} is $\mathrm{NPM}^{(k)}$.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$ and T be a theory. Then the following are equivalent.

- $T \in D$.
- $T^{ind} \in D$.
- $T_P \in D$.

Corollary

- T is NCTP if and only if T^{ind} is NCTP if and only if T_P is NCTP.
- T is NBTP if and only if T^{ind} is NBTP if and only if T_P is NBTP.
- T is NGP if and only if T^{ind} is NGP if and only if T_P is NGP.
- T is NWP if and only if T^{ind} is NWP if and only if T_P is NWP.
- T is $\mathrm{NPM}^{(k)}$ if and only if T^{ind} is $\mathrm{NPM}^{(k)}$ if and only if T_P is $\mathrm{NPM}^{(k)}$.

Application of n-variable theorem

$D_{o-minimality}, D_{\text{distal}} \notin \mathfrak{D}_{n\text{-var}}$ (there is a dense/codense expansion of o-minimal structure, which is not o-minimal and distal).

Application of n-variable theorem

Let $\mathbb{F}$ be a field and $\mathcal{L} \supseteq \mathcal{L}_0 := \{0, +, -, \{\lambda\cdot\}_{\lambda\in\mathbb{F}}\}$. Let T be an $\mathcal{L}$-theory such that $T|_{\mathcal{L}_0}$ is the theory of vector space over $\mathbb{F}$. We assume T has quantifier elimination, eliminates quantifier $\exists^\infty$, and $\mathrm{acl}_T = \mathrm{dcl}_T = \mathrm{span}_{\mathbb{F}}$.

Fix a subring R of $\mathbb{F}$ and let $\hat{R} := \mathrm{Frac}(R)$ and $\mathcal{L}_{R\text{-mod}} := \{0, +, -, \{r\cdot\}_{r\in R}\}$

Let G be a new unary predicate, $P_\phi(\bar{x})$ a new predicate for each $\phi(\bar{x}) \in \mathcal{L}_{R\text{-mod}}$, and $\mathcal{L}_G := \mathcal{L} \cup \{G\} \cup \{P_\phi(\bar{x}) : \phi(\bar{x}) \in \mathcal{L}_{R\text{-mod}}\}$.

Let us consider an $\mathcal{L}_G$-theory $T_R^G \supseteq T$ with the following first-order conditions

- G is dense/codense.
- G is a R-submodule.
- For $\bar{g} \in G$, $P_\phi(\bar{g})$ holds if $\phi(\bar{g})$ holds in the R-module G.
- If $\lambda_0, ..., \lambda_{n-1}$ are $\hat{R}$-linearly independent, then for all $g_0, ..., g_{n-1} \in G$,

$$\sum_{i<n} \lambda_i g_i = 0 \Rightarrow \bigwedge_{i<n} g_i = 0.$$

Remark

If $R \subsetneq \mathbb{F}$, then $(V, G) \models T_R^G$ is not a lovely pair, not a H-structure.

Application of n-variable theorem

Fact [Berenstein, Vassiliev, d'Elbee]

- T is NIP if and only T_R^G is NIP.
- T is stable if and only if T_R^G is stable.
- T is simple if and only if T_R^G is simple.
- T is NTP_1 if and only if T_R^G is NTP_1.
- T is NTP_2 if and only if T_R^G is NTP_2.

Fact [Ahn, K., Lee, Lee]

T is NATP if and only if T_R^G is NATP.

Application of n-variable theorem

Theorem [K.]

Additionally assume that K is a subfield of $\mathbb{F}$. If $D \in \mathfrak{D}_{n\text{-var}}$, then $T \in D$ if and only if $T_K^G \in D$.

Corollary

- T is NCTP if and only T_K^G is NCTP.
- T is NBTP if and only if T_K^G is NBTP.
- T is NGP if and only if T_K^G is NGP.
- T is NWP if and only if T_K^G is NWP.
- T is $\text{NPM}^{(k)}$ if and only if T_K^G is $\text{NPM}^{(k)}$.

Application of n-variable theorem

Definition

Let $\mathbb{M} \models T$ and let $D \subseteq M^n$ be definable (possibly with parameters).
We say that D is *stably embedded* if every subset of D^m
definable in $\mathbb{M}$ is already definable in the induced structure on D
with parameters from D.

Application of n-variable theorem

Let $\mathcal{L} \supseteq \mathcal{L}_{ring}$ and T an $\mathcal{L}$-theory of fields. Let P be a unary predicate and $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. Let ACF_T be an $\mathcal{L}_P$-expansion of ACF satisfying the following.

- P is a subfield of the universe.
- P is a model of T.
- For every n-ary funtion symbol $f \in \mathcal{L} \setminus \mathcal{L}_{ring}$, if $x_0, ..., x_{n-1} \in P$, then $f(x_0, ..., x_{n-1}) \in P$. Else, if some $x_i \notin P$, then we put $f(\bar{x}) = 0$.
- For every n-ary relation symbol $R \in \mathcal{L} \setminus \mathcal{L}_{ring}$, if some $x_i \notin P$, then $\neg R(\bar{x})$.
- The degree of the field extension of the universe over P is infinite. i.e. the universe has infinite dimension as a vector space over P.

Application of n-variable theorem

Fact [d'Elbee, Kaplan, Neuhauser]

- T is NIP if and only if ACF_T is NIP.
- T is stable if and only if ACF_T is stable.
- T is simple if and only if ACF_T is simple.
- T is NTP_1 if and only if ACF_T is NTP_1.

Fact [Ahn, K., Lee, Lee]

- T is NTP_2 if and only if ACF_T is NTP_2.
- T is NATP if and only if ACF_T is NATP.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$. Then $T \in D$ if and only if $ACF_T \in D$.

Corollary

- T is NCTP if and only ACF_T is NCTP.
- T is NBTP if and only if ACF_T is NBTP.
- T is NGP if and only if ACF_T is NGP.
- T is NWP if and only if ACF_T is NWP.
- T is $\text{NPM}^{(k)}$ if and only if ACF_T is $\text{NPM}^{(k)}$.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$. Then $T \in D$ if and only if $ACF_T \in D$.

sketch of proof. It is known that if $c, c' \notin \mathrm{acl}_{\mathcal{L}_{\mathrm{ring}}}(P(\mathbb{M})A)$, then $c \equiv_{P(\mathbb{M})A} c'$. We use this fact. Let $\varphi(x; y)$ witnesses $ACF_T \notin D$ with $(a_\eta)_{\eta \in \mathcal{I}^*}$. Then there exists $b \models \{\varphi(x, a_\eta)\}_{\eta \in X}$ such that b is algebraically independent over $(a_\eta)_{\eta \in X}$. We can decompose b into $b_0 \notin P(\mathbb{M})$ and $b_1 \in P(\mathbb{M})$. Moreover, by the n-variable theorem, we may assume that b_0 is algebraically independent over $P(\mathbb{M})(a_\eta)_{\eta \in X}$. Thus by the fact and the modeling property, we may assume $(a_\eta)_{\eta \in \mathcal{I}^*}$ is indiscernible over b_0. So we may assume that $b \in P(\mathbb{M})$. Now we can apply the stable embeddedness of $P(\mathbb{M})$. We may assume $\varphi \in \mathcal{L}$ and $(a_\eta)_{\eta \in \mathcal{I}^*} \subseteq P(\mathbb{M})$. So φ witnesses $T \notin D$, it is a contradiction.

Application of n-variable theorem

Let $\mathcal{L} \supseteq \mathcal{L}_{ring}$ and T be a complete $\mathcal{L}$-theory of fields with characteristic 0.

Definition

We say T is *algebraically bounded* if $a \in \mathrm{acl}_{\mathcal{L}}(KB) \Leftrightarrow \mathrm{trdeg}(a|K(B)) = 0$ for any $K \prec K^* \models T$, $a \in K^*$, and $B \subseteq K^*$.

Let $\bar{\delta} = (\delta_i)_{i<\lambda}$ be unary function symbols, where λ is a cardinal, and $\mathcal{L}_{\bar{\delta}} := \mathcal{L} \cup \{\delta_i : i < \lambda\}$. For algebraically bounded T, let $T_g^{\bar{\delta},nc}$ be an expansion of T to $\mathcal{L}_{\bar{\delta}}$ such that

- $\delta_i(x+y) = \delta_i(x) + \delta_i(y)$ and $\delta_i(xy) = \delta_i(x)y + x\delta_i(y)$ for each $i < \lambda$,
- for any $k, r < \omega$ and an $\mathcal{L}(A)$-definable set $X \subseteq K^{(k+1)r}$, where $K \models T_g^{\bar{\delta},nc}$ is sufficiently saturated and $A \subseteq K$ is small, if the projection of X onto the first r coordinates has algebraic dimension r over A, then there is $\bar{a} \in K^r$ such that

$$(\bar{a}, \delta_{i_0}(\bar{a}), \ldots, \delta_{i_{k-1}}(\bar{a})) \in X$$

  for any distinct $i_0, \ldots, i_{k-1} < \lambda$.

The derivations are not required to commute.

Application of n-variable theorem

Fact [Fornasiero, Terzo]

- T is stable if and only if $T_g^{\bar{\delta},nc} \in D$ is stable.
- T is NIP if and only if $T_g^{\bar{\delta},nc} \in D$ is NIP.
- T is distal if and only if $T_g^{\bar{\delta},nc} \in D$ is distal.
- T is simple if and only if $T_g^{\bar{\delta},nc} \in D$ is simple.

Fact [Sánchez, Mohamed]

T is NSOP_1 if and only if $T_g^{\bar{\delta},nc} \in D$ is NSOP_1.

Fact [Kaplan, Kesting]

- T is NTP_2 if and only if $T_g^{\bar{\delta},nc} \in D$ is NTP_2, if $|\bar{\delta}| = 1$.
- T is NATP if and only if $T_g^{\bar{\delta},nc} \in D$ is NATP, if $|\bar{\delta}| = 1$.

Application of n-variable theorem

Theorem [K.]

Let $D \in \mathfrak{D}_{n\text{-var}}$. Then $T \in D$ if and only if $T_g^{\bar{\delta},nc} \in D$.

Corollary

- T is NTP_2 if and only $T_g^{\bar{\delta},nc} \in D$ is NTP_2.
- T is NATP if and only $T_g^{\bar{\delta},nc} \in D$ is NATP.
- T is NCTP if and only $T_g^{\bar{\delta},nc} \in D$ is NCTP.
- T is NBTP if and only if $T_g^{\bar{\delta},nc} \in D$ is NBTP.
- T is NGP if and only if $T_g^{\bar{\delta},nc} \in D$ is NGP.
- T is NWP if and only if $T_g^{\bar{\delta},nc} \in D$ is NWP.
- T is $\mathrm{NPM}^{(k)}$ if and only if $T_g^{\bar{\delta},nc} \in D$ is $\mathrm{NPM}^{(k)}$.

Higher-arity

Let $1 < k < \omega$.

- [Shelah] We say T is NIP_k if there do not exist $\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{k-1})$, $(a_i^0)_{i<\omega}, \ldots, (a_i^{k-1})_{i<\omega}$, and $(b_I)_{I \subseteq \omega^k}$ such that
$$\models \varphi(b_I, a^0_{i_0}, \ldots, a^{k-1}_{i_{k-1}}) \text{ if and only if } (i_0, \ldots, i_{k-1}) \in I.$$
- [Terry, Wolf] We say T has FOP_k if there exist $\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{k-1})$, $(a_i^0)_{i<\omega}, \ldots, (a_i^{k-1})_{i<\omega}$, and $(b_f)_{f:\omega^{k-1}\to\omega}$ such that
$$\models \varphi(b_f, a^0_{i_0}, \ldots, a^{k-1}_{i_{k-1}}) \text{ if and only if } i_{k-1} \leq f(i_0, \ldots, i_{k-2}).$$
- [Takeuchi] Let $\mathcal{F}$ be the set of all nondecreasing functions from ω to ω. We say T has OP_2 if there exist $\varphi(\bar{x}, \bar{y}_0, \bar{y}_1)$, $(a_i^0)_{i<\omega}$, $(a_j^1)_{j<\omega}$, and $(b_f)_{f\in\mathcal{F}}$ such that
$$\models \varphi(b_f, a_i^0, a_j^1) \text{ if and only if } j \leq f(i).$$

These are examples of *higher-arity dividing lines (higher-arity stability)*, studied as higher-arity analogues of stability and NIP.
Here, "higher arity" means that the parameter variables part of the witness formula contains multiple blocks $\bar{y}_0, \ldots, \bar{y}_{k-1}$.

Higher-arity

We can do almost the same work for higher-arity dividing lines.

Definition

Let $\mathcal{L}^*$ be a language, $\mathcal{I}^*$ an $\mathcal{L}^*$-structure, $k \in \omega$, $X \subseteq (\mathcal{I}^*)^k$, and $\bar{Y}$ a finite set of subsets of $(\mathcal{I}^*)^k$.
We say a quintuple $(\mathcal{L}^*, \mathcal{I}^*, k, X, \bar{Y})$ satisfies the *n-variable theorem* if

(i) $\mathcal{I}^*$ has the modeling property,

(ii) $\mathcal{I}^*$ is k-weakly monochromatic,

(iii) X is cofinal in $\mathcal{I}^*$,

(iv) $\mathcal{I}^*$ is k-monochromatically extendable over

$$\mathbb{X} := \{\bar{\eta} \in (\mathcal{I}^*)^k : \langle \bar{\eta} \rangle \sim_{\mathcal{L}^*} \langle \bar{\nu} \rangle \text{ for some } \bar{\nu} \in X\},$$

(v) $Y \subseteq \mathbb{X}$ for each $Y \in \bar{Y}$,

(vi) $|Y| > 1$ for each $Y \in \bar{Y}$,

(vii) there exists a disjoint union $W \subseteq (\mathcal{I}^*)^k$ of X' and Y' such that

- $X' \sim^{\text{fin}}_{\mathcal{L}^*} X$ and $Y' \sim^{\text{fin}}_{\mathcal{L}^*} Y$ for some $Y \in \bar{Y}$,
- $\{\bar{\nu}\} \cup X' \sim^{\text{fin}}_{\mathcal{L}^*} X$ for each $\bar{\nu} \in Y'$.

Higher-arity

Definition

For $(\mathcal{L}^*, \mathcal{I}^*, k, X, \bar{Y})$ satisfying n-variable theorem, let $D^{\mathcal{L}^*,\mathcal{I}^*,k}_{X,\bar{Y}}$ be the class of all first-order theories T such that there do not exist $\varphi(x, y_0, ..., y_{k-1})$ in the language of T and $(a_\eta)_{\eta\in\mathcal{I}^*}$ in a model of T such that

(i) $(a_\eta)_{\eta\in\mathcal{I}^*}$ is $\mathcal{I}^*_{\mathcal{L}^*}$-indiscernible,

(ii) $\{\varphi(x, a_{\eta_0}, ..., a_{\eta_{k-1}})\}_{(\eta_0,...,\eta_{k-1})\in X}$ is consistent,

(iii) $\{\varphi(x, a_{\eta_0}, ..., a_{\eta_{k-1}}))\}_{(\eta_0,...,\eta_{k-1})\in Y}$ is inconsistent for each $Y \in \bar{Y}$.

Let $\mathfrak{D}^h_{n\text{-var}}$ be a class of classes of first-order theories such that

$$\mathfrak{D}^h_{n\text{-var}} := \left\{ D^{\mathcal{L}^*,\mathcal{I}^*,k}_{X,\bar{Y}} : (\mathcal{L}^*, \mathcal{I}^*, k, X, \bar{Y}) \text{ satisfies } n\text{-variable theorem} \right\}.$$

Remark

$\mathfrak{D}_{n\text{-var}} \subseteq \mathfrak{D}^h_{n\text{-var}}$.

Higher-arity

Theorem [K.]

If $T \notin D^{\mathcal{L}^*,\mathcal{I}^*,k}_{X,\bar{Y}}$, then there are $\varphi(\bar{x}, \bar{y}_0, ..., \bar{y}_k)$ and $\mathcal{L}^*$-indiscernible $(a_\eta)_{\eta \in \mathcal{I}^*}$ s.t.

- φ and $(a_\eta)_{\eta \in \mathcal{I}^*}$ witness $T \notin D^{\mathcal{L}^*,\mathcal{I}^*,k}_{X,\bar{Y}}$,
- $\dim(\{\varphi(\bar{x}, a_{\eta_0}, ..., a_{\eta_{k-1}})\}_{\eta \in X}) = |\bar{x}|$.

Theorem [K.]

Let $D \in \mathfrak{D}^h_{n\text{-var}}$ and acl $=$ dcl. Then $T \in D$ if and only if $T^{gt} \in D$.

Theorem [K.]

Let $D \in \mathfrak{D}^h_{n\text{-var}}$. Then $T \in D$ if and only if $T_P \in D$ if and only if $T^{ind} \in D$.

Theorem [K.]

Let $D \in \mathfrak{D}^h_{n\text{-var}}$. Then $T \in D$ if and only if $T^G_K \in D$.

Theorem [K.]

Let $D \in \mathfrak{D}^h_{n\text{-var}}$. Then $T \in D$ if and only if $T^{\bar{\delta},nc}_g \in D$.

Higher-arity

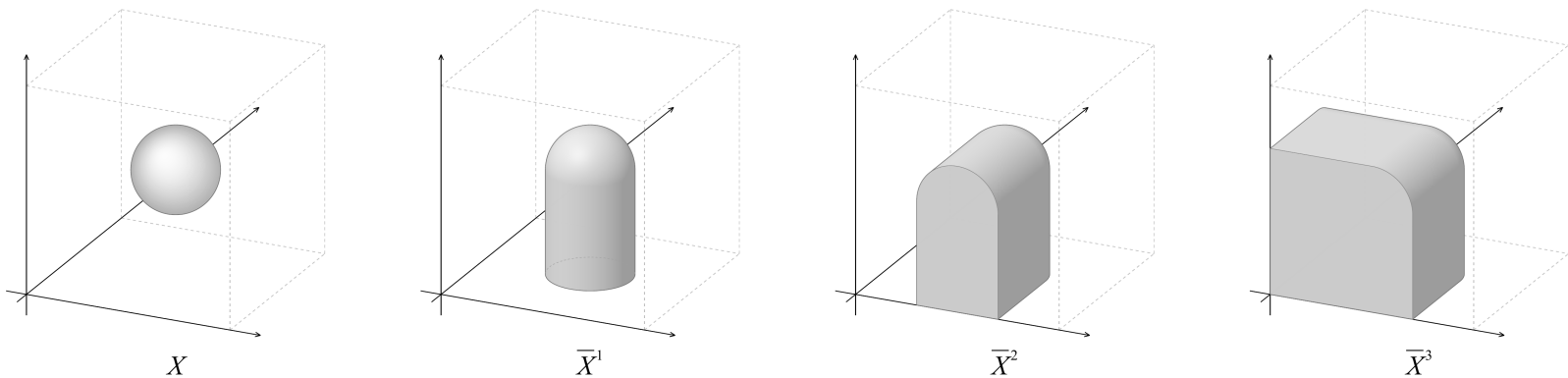
In fact, IP_k, FOP_k, and OP_2 can all be described within a single framework.

Notation

Let $0 < k < \omega$, $0 \leq m \leq k$, and $X \subseteq \omega^k$. The *m-th down closure* of X is

$$\overline{X}^m := \left\{ (i_0, \ldots, i_{k-1}) \in \omega^k : \begin{array}{l} \text{there exists } (j_0, \ldots, j_{k-1}) \in X \\ \text{such that } i_r = j_r \text{ for } r < k - m, \\ \text{and } i_r \leq j_r \text{ for } k - m \leq r < k \end{array} \right\}.$$

The first $k - m$ coordinates stay fixed, and the last m coordinates may decrease.



X and its l-th down closures in $(0, 1)^3$

Higher-arity

Definition

Let $0 < k < \omega$ and $0 \leq l \leq k$. We say a complete theory T has $\frac{l}{k}P_k$ if there exist a formula $\varphi(x, y_0, \ldots, y_{k-1})$ and sequences $(a_i^0)_{i<\omega}, \ldots, (a_i^{k-1})_{i<\omega}$ such that, for every $X \subseteq \omega^k$, there exists b_X satisfying

$$\models \varphi(b_X, a^0_{i_0}, \ldots, a^{k-1}_{i_{k-1}}) \text{ if and only if } (i_0, \ldots, i_{k-1}) \in X^{k-l}$$

for all $(i_0, \ldots, i_{k-1}) \in \omega^k$. We say T is $\frac{l}{k}NP_k$ if it does not have $\frac{l}{k}\mathrm{P}_k$.

Remark/Definition

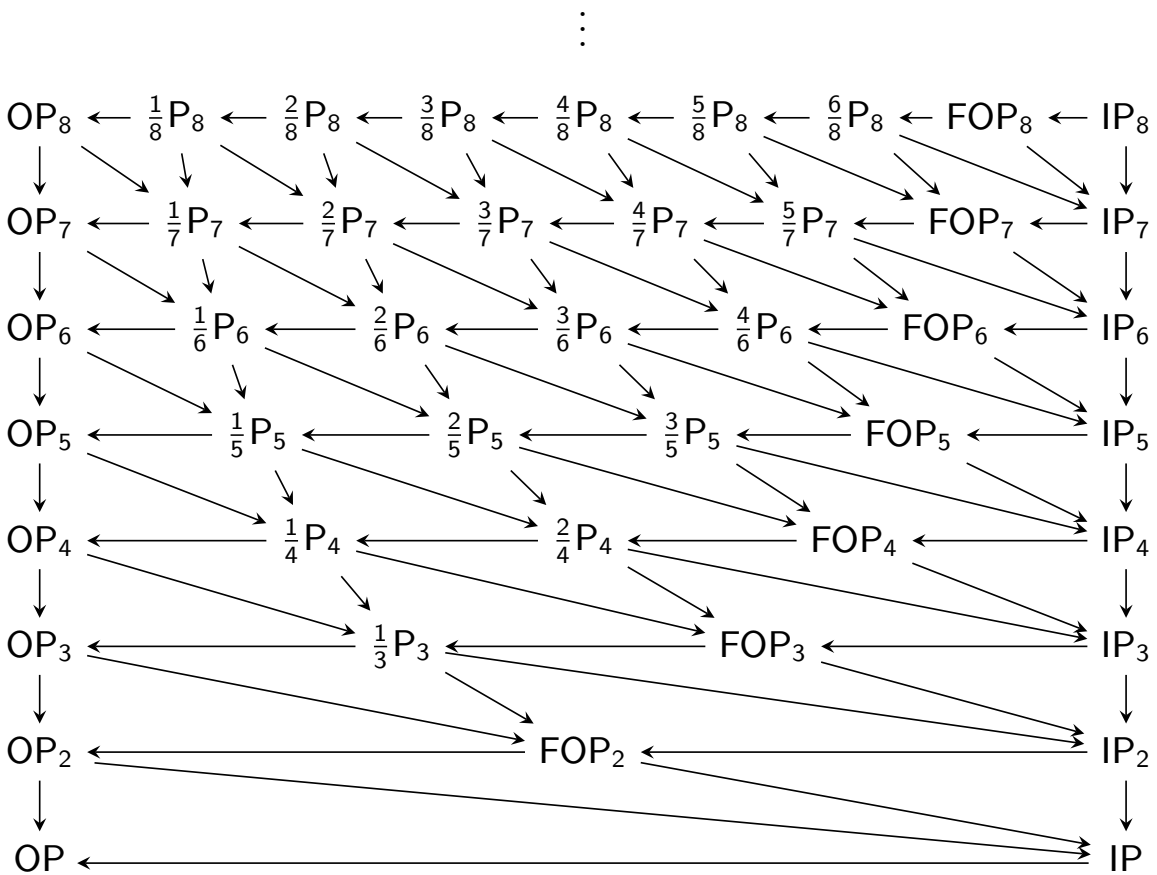
$$\frac{0}{2}\mathrm{P}_2 = \mathrm{OP}_2, \quad \frac{k-1}{k}\mathrm{P}_k = \mathrm{FOP}_k, \quad \frac{k}{k}\mathrm{P}_k = \mathrm{IP}_k.$$

And in this sense, we can define OP_k to be $\frac{0}{k}\mathrm{P}_k$.

Higher-arity

Proposition [K.]

- If T has $\frac{l+1}{k}\mathrm{P}_k$, then it has $\frac{l}{k}\mathrm{P}_k$.
- If T has $\frac{l}{k+1}\mathrm{P}_k + 1$, then it has $\frac{l+1}{k}\mathrm{P}_k$.



Higher-arity

Theorem [K.]

Let $D_{\frac{l}{k}\mathrm{NP}_k}$ be the class of complete $\frac{l}{k}\mathrm{NP}_k$ theories. Then $D_{\frac{l}{k}\mathrm{NP}_k} \in \mathfrak{D}^h_{n\text{-var}}$ for $1 < k < \omega$ and $0 \leq l \leq k$.

Corollary

- T is NOP_k if and only if T^{gt} is NOP_k.
- T is NFOP_k if and only if T^{gt} is NFOP_k.
- T is NIP_k if and only if T^{gt} is NIP_k.
- T is NOP_k if and only if T_P is NOP_k if and only if T^{ind} is NOP_k.
- T is NFOP_k if and only if T_P is NFOP_k if and only if T^{ind} is NFOP_k.
- T is NIP_k if and only if T_P is NIP_k if and only if T^{ind} is NIP_k.
- T is NOP_k if and only if T^G_K is NOP_k.
- T is NFOP_k if and only if T^G_K is NFOP_k.
- T is NIP_k if and only if T^G_K is NIP_k.
- T is NOP_k if and only if $T_g^{\bar{\delta},nc}$ is NOP_k.
- T is NFOP_k if and only if $T_g^{\bar{\delta},nc}$ is NFOP_k.
- T is NIP_k if and only if $T_g^{\bar{\delta},nc}$ is NIP_k.

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