On the Hierarchy between SOP_1 and SOP_2

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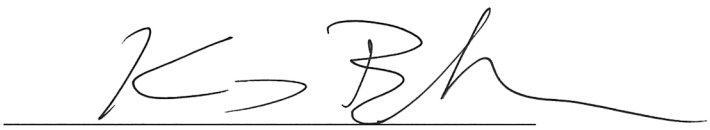
On the Hierarchy between SOP_1 and SOP_2

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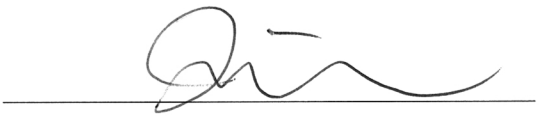
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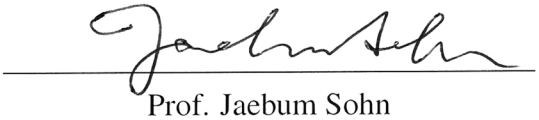
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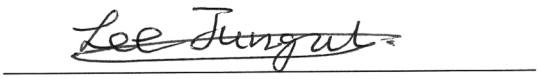
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ABSTRACT

On the Hierarchy between SOP_1 and SOP_2

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In this thesis, we study model-theoretic tree properties: SOP_1 and SOP_2. First, we introduce a notion of tree-indiscernibility which is called β-indiscernibility, and show that the existence of a witness of SOP_1 implies the existence of a β-indiscernible witness of SOP_1. As an application, we show that if there is an SOP_1 theory which is not SOP_2, then the theory has the antichain tree property (ATP), which has the opposite nature of SOP_2 in some combinatoric sense. We also show that having ATP implies SOP_1 and TP_2.

And we define a new notion of tree-property which is called 1-strictly strong order property (SSOP_1). SSOP_1 is a special case of SOP_1 and we show that the existence of a witness of SSOP_1 implies the existence of a β-indiscernible witness of SSOP_1, as in the

case of SOP_1. And we also prove that having SSOP_1 and having ATP are equivalent.

It is well known that having SOP_2 implies having SOP_1, but the converse is still open. Our results reduce the question which asks if SOP_1 and SOP_2 are the same to the question which asks if having ATP or SSOP_1 implies having SOP_2. In other words, if we want to conclude that a theory has SOP_2 from the assumption that the theory has SOP_1, we may assume that the theory has ATP and SSOP_1, which are stronger conditions than SOP_1.

We construct an example of a theory which has ATP. This theory has a quantifier free formula φ which witnesses ATP, and no quantifier free formula witness SOP_2. In particular, no conjunction of φ witness SOP_2. This example shows that SOP_1 and SOP_2 are distinct at the level of formulas.

At the end of this thesis, we develop some combinatorial techniques handling antichains. And by using these techniques we show that if a theory has ATP, then the theory has a witness of ATP in a single free variable.

Key words : Tree property, SOP_1, SOP_2, SSOP_1, Indiscernible trees, Antichain tree property, One-variable theorem

Chapter 1

Introduction

This study contains some results in model theory, which is related with classification theory. Since Džamonja and Shelah introduced the notions of SOP_1 and SOP_2 in [4], it is well known that if a theory has SOP_2 then it has SOP_1, but it is still an open question asking if the converse holds. We observe some interesting phenomenon under the assumption that SOP_1 and SOP_2 are not the same. In this thesis, we introduce a new notion of tree property which is called antichain tree property (ATP) having some opposite nature of SOP_2, and we show that if a theory has SOP_1 and does not have SOP_2, then the theory always has ATP. And by constructing of a theory by using ATP, we show that in fact SOP_1 and SOP_2 are distinct at the level of formulas. These result can be a partial answer to the question asking if SOP_1 and SOP_2 are the same.

Since SOP_1 and SOP_2 are witnessed by a set of parameters which forms a binary tree, tree-indiscernibility can be a useful tool of studying SOP_1 and SOP_2. In [8], [9], [15], it is proved that s-, ss-, and str-indiscernibility have the modeling property, which means for any given binary tree of paremeters $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$, there exists s-, ss-, str-indiscernible binary trees of parameters which are based on $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$. But in some cases of investigating SOP_1, the modeling property of those indiscernibility are not so suitable

because to apply the modeling property on a tree, the tree should consist of infinitely many branches for each node. But a witness of SOP_1 is defined by a binary tree. Even if we extend a witness of SOP_1 to an infinitary tree, whose all nodes have infinitely many branches, and apply the modeling property of ss- and str-indiscernibility, it is not sure the results still witness SOP_1. And even if the result witnesses SOP_1, str-indiscernibility forces the result to witness SOP_2. On the other hand, if we apply the modeling property of s-indiscernibility to an infinitary tree which is extended from a witness of SOP_1, we can obtain an s-indiscernible witness of SOP_1 but its homogeneity is not strong enough for some purpose. We will discuss this more in details in Chapter 2.

Thus we study a suitable tree indiscernibility for witnesses of SOP_1 and we call it β-indiscernibility. We show that if a witness of SOP_1 is given, then we can find a β-indiscernible witness of SOP_1. Unlike ss- and str-indiscernibility, β-indiscernibility preserves SOP_1 and does not force a witness of SOP_1 to be a witness of SOP_2. And it has stronger homogeneity than s-indiscernibility because it does not distinguish nodes at different levels. β-indiscernibility plays an important role when we construct an antichain tree in SOP_1-NSOP_2 theories.

In this thesis, we also introduce a new notion of tree properties which is called 1-strictly strong order property (SSOP_1). Roughly speaking, the definition of SSOP_1 is obtained from the definition of SOP_1, by adding consistency condition where the definition of SOP_2 takes inconsistency condition, while the definition of ATP is obtained from the definition of SOP_2 by interchanging consistency and inconsistency conditions of it. We will describe them more precisely in Chapter 3 and 4. And we will observe that for any theory, having ATP and having SSOP_1 are equivalent.

Thus these results reduce the question asking if SOP_1 and SOP_2 are the same to the question asking if having ATP or SSOP_1 always implies having SOP_2. If SOP_1 and SOP_2 are not the same, then there exists an SOP_1 theory which is not SOP_2. But by what we will show in Chapter 3 and 4, the theory has ATP and SSOP_1. Thus if we want to show SOP_1 and SOP_2 are the same, it is enough to show that for any theory, having ATP or SSOP_1 always implies having SOP_2.

As we mentioned in the abstract, in Chapter 5 we construct an example of ATP theory whose witness does not witness SOP_2 and every quantifier free formula does not witness SOP_2 modulo the theory. More precisely, we construct a structure $\mathbb{C}$ on a relational language $\mathcal{L}$ which consists of one binary relation symbol $R(x, y)$. In the theory $\mathrm{Th}(\mathbb{C})$, the formula $R(x, y)$ witnesses ATP, and every quantifier free formula does not witness SOP_2. Hence we can say that SOP_1 and SOP_2 are distinct at the level of formulas.

In the last chapter, we show that every ATP theory has a witness of ATP in a single free variable. To prove this we find an indiscernibility for antichains, and establish some lemmas which are antichain versions of "Path-Collapse Lemma" in [3], allow us to reduce the number of free variables in a witness of ATP.

Recently, there have been many studies on NSOP_1 theories. If it turns out that SOP_1 and SOP_2 are equivalent in the level of theories, then the results on NSOP_1 can be extended to NSOP_2 and NTP_1 theories as it is well-known that SOP_2 and TP_1 are equivalent. Even if it turns out that SOP_1 and SOP_2 are distinct, we expect that the difference between SOP_1 and SOP_2 will be an useful clue to studying NSOP_1 theories. Results in this thesis can be understood as attempts to contribute to this topic.

1.1 Preliminaries on basic model theory

The following preliminaries on basic model theory are from [5], [16], [6], and [11].

Definition 1.1. By a language, we mean a set of constant symbols, relation symbols, and function symbols. Let $\mathcal{L}$ be a language and

$$\mathcal{L} = \{c_0, ..., c_n, R_0(x_0, ..., x_{d_0}), ..., R_m(x_0, ..., x_{d_m}), f_0(x_0, ..., x_{e_0}), ..., f_l(x_0, ..., x_{e_l})\}$$

, where c_i, R_j, f_k are constant symbol, relation symbol, and function symbol respectively, for each i, j, k. We assume that we have given the *logical symbols* such as $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\exists$, $\forall$, which mean 'and', 'or', 'implies', 'not', 'there exists', 'for all', respectively. We also assume that the equality symbol $=$ and the variable symbols x, y, ... , x_0, x_1, ... are given.

(i) By an $\mathcal{L}$-*structure*, say $\mathbb{M}$, we mean a tuple $(M; c_0^{\mathbb{M}}, ..., c_n^{\mathbb{M}}; R_0^{\mathbb{M}}, ..., R_m^{\mathbb{M}}; f_0^{\mathbb{M}}, ..., f_l^{\mathbb{M}})$, where M is a set, $c_i^{\mathbb{M}} \in M$ for all $i \leq n$, $R_j^{\mathbb{M}} \subseteq M^{d_j}$ for all $j \leq m$, and $f_k : M^{e_k} \rightarrow M$ for all $k \leq l$. M is said to be the universe of $\mathbb{M}$.

(ii) We define $\mathcal{L}$-*terms* recursively as follows

- every constant is an $\mathcal{L}$-term,
- every variable is an $\mathcal{L}$-term,
- if $t_0, ..., t_n$ are $\mathcal{L}$-terms and $f(x_0, ..., x_n) \in \mathcal{L}$ is a function symbol, then $f(t_0, ..., t_n)$ is an $\mathcal{L}$-term.

(iii) We define $\mathcal{L}$-*formula* recursively as follows

- if $t_0, ..., t_n$ are $\mathcal{L}$-terms and $R(x_0, ..., x_n) \in \mathcal{L}$ is a relation symbol, then $R(t_0, ..., t_n)$ is an $\mathcal{L}$-formula,

- if φ and ψ are $\mathcal{L}$-formulas, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$ and $\neg\varphi$ are $\mathcal{L}$-formulas,
- if φ is an $\mathcal{L}$-formula and x is a variable, then $\exists x\varphi$ and $\forall x\varphi$ are $\mathcal{L}$-formulas.

(iv) Let φ be an $\mathcal{L}$ formula. By a *free variable* of φ we mean a variable x, which occurs in φ, is not captured by a quantifier $\exists$ or $\forall$.

(v) By an $\mathcal{L}$-sentence we mean an $\mathcal{L}$-formula which has no free variable.

(vi) By an *$\mathcal{L}$-theory*, we mean a set of $\mathcal{L}$-sentences. We say a consistent theory T is complete if for any sentence σ, either $\sigma \in T$ or $\neg\sigma \in T$.

(vii) By a *type*, we mean a set of formula with free variables which is consistent.

(viii) Let $\mathbb{M}$, σ, and $\varphi(x)$ be an $\mathcal{L}$-structure, an $\mathcal{L}$-sentence, and an $\mathcal{L}$-formula. By $\mathbb{M} \models \sigma$, we mean that σ is true in $\mathbb{M}$ and by $\mathbb{M} \models \varphi(m)$, we mean $\varphi(x)$ is true in $\mathbb{M}$ with a solution $m \in M$.

(ix) Let $\mathbb{M}$ and Σ be an $\mathcal{L}$-structure and an $\mathcal{L}$-theory. By $\mathbb{M} \models \Sigma$ we mean $\mathbb{M} \models \sigma$ for all $\sigma \in \Sigma$.

(x) Let $\mathbb{M}$ be an $\mathcal{L}$-structure. By $\mathrm{Th}(\mathbb{M})$ we mean the set of all $\mathcal{L}$-sentences which are true in $\mathbb{M}$.

In many cases, for a given $\mathcal{L}$-stucture $\mathbb{M}$, we can regard an element of M as a constant. For $A \subseteq M$, by $\mathcal{L}(A)$ we mean a language obtained from $\mathcal{L}$ by adding A as a set of constants. If we treat elements of the structure like constants, we call them parameters.

Definition 1.2. Let $\mathcal{L}$ be a language and $\mathbb{M}$ be an $\mathcal{L}$-structure.

(i) For $A \subseteq M$ and $b \in M$, by $\mathrm{tp}(b/A)$ we mean the set of $\mathcal{L}(A)$-formula $\varphi(x)$ which satisfies $\mathbb{M} \models \varphi(m)$.

(ii) By $a_0, ..., a_n \equiv_A b_0, ..., b_n$ we mean $\text{tp}(a_0, ..., a_n/A) = \text{tp}(b_0, ..., b_n/A)$.

(iii) A type $\Gamma(x)$ is said to be *satisfiable* in $\mathbb{M}$ if there is $m \in M$ such that $\mathbb{M} \models \varphi(m)$ for all $\varphi(x) \in \Gamma(x)$. A type $\Gamma(x)$ is said to be *finitely satisfiable* if every finite subset $\Gamma_0(x) \subseteq \Gamma(x)$ is satisfiable.

Fact 1.3. (Compactness Theorem) *If $\Gamma(x)$ is finitely satisfiable, then it is consistent.*

Definition 1.4. Let $\mathcal{L}$ be a language, $\mathbb{M}$ an $\mathcal{L}$-structure, and κ an infinite cardinal number. We say $\mathbb{M}$ is *κ-saturated* if for all $A \subseteq M$ with $|A| < \kappa$, every type of $\mathcal{L}(A)$ is realized by some element of M.

Definition 1.5. Let $\mathcal{L}$ be a language, $\mathbb{M}$ be an $\mathcal{L}$-structure, $\overline{m} = \langle m_i \rangle_{i \in \omega}$ be a sequence of elements of M. We say $\overline{m}$ is *(order-)indiscernible* if for any $\varphi(\overline{x}) = \varphi(x_0, ..., x_n) \in \mathcal{L}$, $i_0 < \cdots < i_n$ and $j_0 < \cdots < j_n$ in ω,

$$\mathbb{M} \models \varphi(m_{i_0}, ..., m_{i_n}) \text{ if and only if } \mathbb{M} \models \varphi(m_{j_0}, ..., m_{j_n}).$$

Definition 1.6. Let $\mathcal{L}$ be a language, T be an $\mathcal{L}$-theory.

(i) We say an $\mathcal{L}$-formula φ is *quantifier free* if the formula does not contain quantifier.

(ii) Let σ be an $\mathcal{L}$-sentence. By $T \models \sigma$, we mean that for every $\mathcal{L}$-structure $\mathbb{M}$, if $\mathbb{M} \models T$, then $\mathbb{M} \models \sigma$.

(iii) We say T has *quantifier elimination* if for all $\mathcal{L}$-formula $\varphi(x)$, there exists a quantifier free formula $\psi(x)$ such that $T \models \forall x(\varphi(x) \leftrightarrow \psi(x))$.

In many model theoretic study, it is assumed that there exists a κ-saturated model where κ is sufficiently large, hence that every model appears in the study can be embedded into that κ-saturated model. We call it a *monster model*, and many models in the

discussion can be regarded as elementary submodels of it. If there is no confusion in the context, we omit the monster model $\mathbb{M}$ in the expression "$\mathbb{M} \models \sigma$", so that we just write "$\models \sigma$".

1.2 Preliminaries on tree properties

The following preliminaries on tree properties are from [4], [2], [9], [3], [7], [14], and [8]. Let $\mathcal{L}$ be a language, T be a complete $\mathcal{L}$-theory, and $\mathbb{M}$ be a monster model of T.

Notation 1.7. Let κ and λ be cardinals.

(i) By κ^λ we mean the set of all functions from λ to κ.

(ii) By $\kappa^{<\lambda}$ we mean $\bigcup_{\alpha<\lambda} \kappa^\alpha$ and call it a tree. If $\kappa = 2$, we call it a binary tree. If $\kappa \geq \omega$, then we call it an infinitary tree.

(iii) By $\emptyset$ or $\langle\rangle$, we mean the empty string in $\kappa^{<\lambda}$, which means the empty set (recall that every function can be regarded as a set of ordered pairs).

Let $\eta, \nu \in \kappa^{<\lambda}$.

(iv) By $\eta \trianglelefteq \nu$ we mean $\eta \subseteq \nu$. If $\eta \trianglelefteq \nu$ or $\nu \trianglelefteq \eta$, then we say η and ν are comparable.

(v) By $\eta \perp \nu$ we mean that $\eta \not\trianglelefteq \nu$ and $\nu \not\trianglelefteq \eta$. We say η and ν are incomparable if $\eta \perp \nu$.

(vi) By $\eta \wedge \nu$ we mean the maximal $\xi \in \kappa^{<\lambda}$ such that $\xi \trianglelefteq \eta$ and $\xi \trianglelefteq \nu$.

(vii) By $l(\eta)$ we mean the domain of η.

(viii) By $\eta <_{len} \nu$ we mean $l(\eta) < l(\nu)$.

(ix) By $\eta <_{lex} \nu$ we mean that $\eta \trianglelefteq \nu$, or $\eta \perp \nu$ and $\eta(l(\eta \wedge \nu)) < \nu(l(\eta \wedge \nu))$.

(x) By $\eta^\frown \nu$ we mean $\eta \cup \{(i + l(\eta), \nu(i)) : i < l(\nu)\}$.

Let $i, i_0, ..., i_{n-1} \in \kappa$.

(xi) By $\langle i_0 ... i_{n-1} \rangle$ we mean the function $\zeta \in \kappa^n$ such that $\zeta(k) = i_k$ for all $k \in n$. Sometimes we just write it $i_0 ... i_{n-1}$ if there is no confusion.

(xii) By $\langle i^{(n)} \rangle$ we mean the function $\zeta \in \kappa^n$ such that $\zeta(k) = i$ for all $k \in n$. Sometimes we just write it $i^{(n)}$ or i^n if there is no confusion. $\langle i^{(0)} \rangle$ is defined by $\langle \rangle$.

Definition 1.8. Let $\varphi(x, y)$ be an $\mathcal{L}$-formula.

(i) $\varphi(x, y)$ is said to be having the *order property* (OP) if there is a sequence $\langle a_i, b_i \rangle_{i \in \omega}$ whose components are from the monster model, which satisfies

$$\models \varphi(a_i, b_j) \text{ if and only if } i < j$$

for all $i, j \in \omega$.

Let $k \in \omega$. A type $\Gamma(x)$ is said to be k-inconsistent if every subset $\Gamma_0(x) \subseteq \Gamma(x)$ with $|\Gamma_0| \geq k$ is inconsistent.

(ii) $\varphi(x, y)$ is said to be having the *tree property* (TP) if there exists a tree-indexed set $\langle a_\eta \rangle_{\eta \in \omega^{<\omega}}$ of parameters and $k \in \omega$ such that

$\{\varphi(x, a_{\eta \restriction n})\}_{n \in \omega}$ is consistent for all $\eta \in \omega^\omega$ (path consistency),

$\{\varphi(x, a_{\eta^\frown i})\}_{i \in \omega}$ is k-inconsistent for all $\eta \in \omega^{<\omega}$.

(iii) $\varphi(x, y)$ is said to be having the *tree property of the first kind* (TP_1) if there is a tree-indexed set $\langle a_\eta \rangle_{\eta \in \omega^{<\omega}}$ of parameters such that

$\{\varphi(x, a_{\eta \restriction n})\}_{n \in \omega}$ is consistent for all $\eta \in \omega^\omega$,

$\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu$.

(iv) $\varphi(x, y)$ is said to be having the *tree property of the second kind* (TP_2) if there is an array-indexed $(a_{i,j})_{i,j \in \omega}$ of parameters such that

$\{\varphi(x, a_{n,\eta(n)})\}_{n \in \omega}$ is consistent for all $\eta \in \omega^\omega$,

$\{\varphi(x, a_{i,j}), \varphi(x, a_{i,k}))\}$ is inconsistent for all $i, j, k \in \omega$ with $j \neq k$.

(v) $\varphi(x, y)$ is said to be having the *1-strong order property* (SOP_1) if there is a binary-tree-indexed $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ of parameters such that

$\{\varphi(x, a_{\eta \restriction n})\}_{n \in \omega}$ is consistent for all $\eta \in 2^\omega$,

$\{\varphi(x, a_{\eta^\frown 1}), \varphi(x, a_{\eta^\frown 0^\frown \nu})\}$ is inconsistent for all $\eta, \nu \in 2^{<\omega}$.

(vi) $\varphi(x, y)$ is said to be having the *2-strong order property* (SOP_2) if there is a binary-tree-indexed $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ of parameters such that

$\{\varphi(x, a_{\eta \restriction n})\}_{n \in \omega}$ is consistent for all $\eta \in 2^\omega$,

$\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu$.

(vii) For $n > 2$, $\varphi(x, y)$ is said to be having the *n-strong order property* (SOP_n) if there is a sequence $\langle a_i \rangle_{i \in \omega}$ of parameters such that

$\models \varphi(a_i, a_j)$ for all $i < j$,

and there is no $m_0, ..., m_{n-1} \in M$ such that

$$\models \varphi(m_0, m_1) \wedge ... \wedge \varphi(m_{n-2}, m_{n-1}) \wedge \varphi(m_{n-1}, m_0).$$

(viii) $\varphi(x, y)$ is said to be having the *strong order property* (SOP) if there is a sequence $\langle a_i \rangle_{i \in \omega}$ of parameters such that

$$\models \varphi(a_i, a_j) \text{ for all } i < j,$$

and for all $n > 2$ there is no $m_0, ..., m_{n-1} \in M$ such that

$$\models \varphi(m_0, m_1) \wedge ... \wedge \varphi(m_{n-2}, m_{n-1}) \wedge \varphi(m_{n-1}, m_0).$$

(ix) We say a theory has OP if there is a formula having OP with respect to its monster model of the theory. Sometimes we say that the theory is OP, and we call the theory an OP theory. We define TP theory, TP_1 theory, TP_2 theory, SOP_1 theory, SOP_2 theory, SOP_n theory for $n > 2$, and SOP theory in the same manner.

(x) We say a theory is NOP if the theory is not OP, and we call the theory NOP theory. We define NTP theory, NTP_1 theory, NTP_2 theory, NSOP_1 theory, NSOP_2 theory, NSOP_n theory for $n > 2$, and NSOP theory in the same manner.

In fact, NOP theories are more widely known as *stable* theories, and NTP theories are also known as *simple* theories.

Fact 1.9. *(i) If a theory has TP, then it has OP.*

(ii) A theory has TP if and only if it has TP_1 or TP_2.

(iii) For $n > 0$, if a theory has SOP_{n+1}, then the theory has SOP_n.

(iv) If a theory has SOP, then the theory has SOP_n for all $n > 0$.

(v) If a theory has SOP_1, then the theory has TP.

(vi) A theory has SOP_2 if and only if the theory has TP_1.

Thus we have the following diagram.

$$\begin{array}{ccccccc} \mathrm{SOP} \Rightarrow \cdots \Rightarrow \mathrm{SOP}_{n+1} \Rightarrow \mathrm{SOP}_n \Rightarrow \cdots \Rightarrow & \mathrm{SOP}_2 & \Rightarrow & \mathrm{SOP}_1 & & \\ & \Updownarrow & & \Downarrow & & \\ & \mathrm{TP}_1 & \Rightarrow & \mathrm{TP} & \Leftarrow & \mathrm{TP}_2 \\ & & & \Updownarrow & & \\ & & & \text{non-simple} & & \\ & & & \Downarrow & & \\ & & & \text{unstable} & & \end{array}$$

It is known that every oneside implication in the diagram is proper, except the implication from SOP_3 to SOP_2 and the implication from SOP_2 to SOP_1. For these two implications, we do not know whether their converse hold or not.

Džamonja and Shelah's claim can be a good clue to understand the question asking if implication from SOP_2 to SOP_1 is proper. They claim that

Claim 1.10. [4, Claim 2.19] Suppose that $\varphi(x, y)$ satisfies SOP_1, but there is no $n \in \omega$ such that the formula $\varphi_n(x, y_0, ..., y_{n-1}) = \bigwedge_{k<n} \varphi(x, y_k)$ satisfies SOP_2. Then there is a witness $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ for $\varphi(x, y)$ satisfying SOP_1 which in addition satisfies

(i) if $X \subseteq 2^{<\omega}$, and there are no $\eta, \nu \in X$ such that $\eta^\frown 0 \trianglelefteq \nu$, then $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent,

(ii) $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ is 3-fbti.

In fact the statement is not true. Although the statement is not true, their claim gives insight to the difference between SOP_1 and SOP_2. We modify the statement to be more adequate as follows.

Proposition 4.5. *Suppose that $\varphi(x, y)$ satisfies SOP_1, but there is no $n \in \omega$ such that the formula $\varphi_n(x, y_0, ..., y_{n-1}) = \bigwedge_{k<n} \varphi(x, y_k)$ satisfy SOP_2. Then there is a witness $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ for $\varphi(x, y)$ satisfying SOP_1 which in addition satisfies*

(i) $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ witnesses SSOP_1 with φ,

(ii) $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ is β-indiscernible,

and we prove it. We also show that if a formula has ATP, then it has SSOP_1, and vice versa. Thus if there is a formula having SOP_1 and every finite conjunction of the witness does not witness SOP_2, then the formula has ATP and SSOP_1. Thus every SOP_1-NSOP_2 theory has ATP and SSOP_1 if such a theory exists.

Chapter 2

Tree indiscernibility for witnesses of $\mathbf{SOP}_1$

2.1 Modeling property of tree indiscernibility

Let $\mathcal{L}_{str} = \{\trianglelefteq, <_{lex}, \wedge\}$, $\mathcal{L}_{ss} = \mathcal{L}_{str} \cup \{<_{len}\}$, and $\mathcal{L}_s^\alpha = \mathcal{L}_{ss} \cup \{P_\beta\}_{\beta<\alpha}$ be languages where $\trianglelefteq, <_{lex}, <_{len}$ are binary relation symbols, $\wedge$ is a binary function symbol, and P_β is an unary relation symbol for each $\beta < \alpha$. Then for cardinals κ and λ with $\lambda \geq \alpha$, a tree $\langle a_\eta \rangle_{\eta \in \kappa^{<\lambda}}$ can be regarded as an $\mathcal{L}_{str}$-structure and an $\mathcal{L}_{ss}$-structure whose interpretations of $\trianglelefteq, <_{lex}, \wedge, <_{len}$ follow Notation 1.7. If we interpret P_β as the set of elements of $\kappa^{<\lambda}$ whose length is β, then the tree can be regarded as an $\mathcal{L}_s^\alpha$-structure. If there is no confusion, we omit α in $\mathcal{L}_s^\alpha$ and write it $\mathcal{L}_s$.

Remark 2.1. The s-, ss-, and str- before the word 'indiscernibility' mean Shelah-, strong Shelah-, and strong-.

Definition 2.2. Let $\overline{\eta} = \langle \eta_0, ..., \eta_n \rangle$ and $\overline{\nu} = \langle \nu_0, ..., \nu_n \rangle$ be finite tuples of $\kappa^{<\lambda}$.

(i) By $\mathrm{qf}_{str}(\overline{\eta})$ we mean the set of quantifier free $\mathcal{L}_{str}$-formulas $\varphi(\overline{x})$ such that $\kappa^{<\lambda} \models \varphi(\overline{\eta})$. We define $\mathrm{qf}_{ss}(\overline{\eta})$ and $\mathrm{qf}_s(\overline{\eta})$ in the same manner.

(ii) By $\overline{\eta} \sim_{str} \overline{\nu}$ we mean $\mathrm{qf}_{str}(\overline{\eta}) = \mathrm{qf}_{str}(\overline{\nu})$. By the same manner, we define $\overline{\eta} \sim_{ss} \overline{\nu}$ and $\overline{\eta} \sim_{s} \overline{\nu}$.

Let $\mathcal{L}$ be a language, T be a complete $\mathcal{L}$-theory, $\mathbb{M}$ be a monster model of T, and $\langle a_\eta \rangle_{\eta \in \kappa^{<\lambda}}$, $\langle b_\eta \rangle_{\eta \in \kappa^{<\lambda}}$ tree-indexed sets of parameters from M. For $\overline{\eta} = \langle \eta_0, ..., \eta_n \rangle$, we write $\langle a_{\eta_0}, ..., a_{\eta_n} \rangle$ shorten to $\overline{a}_{\overline{\eta}}$. For any finite set of $\mathcal{L}$-formulas Δ and a type Γ, we write $\{\varphi \in \Gamma : \varphi \in \Delta\}$ shorten to Γ_Δ. By $\overline{a}_{\overline{\eta}} \equiv_{\Delta, A} \overline{b}_{\overline{\nu}}$ we mean $\mathrm{tp}_\Delta(\overline{a}_{\overline{\eta}}/A) = \mathrm{tp}_\Delta(\overline{b}_{\overline{\nu}}/A)$.

(iii) We say $\langle a_\eta \rangle_{\eta \in \kappa^{<\lambda}}$ is str-indiscernible if $\mathrm{tp}(\overline{a}_{\overline{\eta}}) = \mathrm{tp}(\overline{a}_{\overline{\nu}})$ for all $\mathrm{qf}_{str}(\overline{\eta}) = \mathrm{qf}_{str}(\overline{\nu})$. By the same manner we define ss-indiscernibility and s-indiscernibility.

(iv) We say $\langle b_\eta \rangle_{\eta \in \kappa^{<\lambda}}$ is str-based on $\langle a_\eta \rangle_{\eta \in \kappa^{<\lambda}}$ if for all $\overline{\eta}$ and a finite set of $\mathcal{L}$-formulas Δ, there is $\overline{\nu}$ such that $\overline{\eta} \sim_{str} \overline{\nu}$ and $\overline{b}_{\overline{\eta}} \equiv_\Delta \overline{a}_{\overline{\nu}}$. By the same manner we define ss-basedness and s-basedness.

Fact 2.3. *Let a tree-indexed set $\langle a_\eta \rangle_{\eta \in \omega^{<\omega}}$ be given. Then*

(i) There is an s-indiscernible $\langle b_\eta \rangle_{\eta \in \omega^{<\omega}}$ which is s-based on $\langle a_\eta \rangle_{\eta \in \omega^{<\omega}}$.

(ii) There is an ss-indiscernible $\langle b_\eta \rangle_{\eta \in \omega^{<\omega}}$ which is ss-based on $\langle a_\eta \rangle_{\eta \in \omega^{<\omega}}$.

(ii) There is a str-indiscernible $\langle b_\eta \rangle_{\eta \in \omega^{<\omega}}$ which is str-based on $\langle a_\eta \rangle_{\eta \in \omega^{<\omega}}$.

The proof can be found in [9], [8], and [15]. The above statements are called the modeling property of each indiscernibility.

Now we explain why it is difficult to obtain an indiscernible witness of SOP_1 by applying the modeling properties of ss- and str-indiscernibility. First, to apply those modeling properties, the witness should be an infinitary tree. But a witness of SOP_1 is

defined to be a binary tree, thus we extend the witness of SOP_1, say $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$, to an infinitary tree $\langle b_\eta \rangle_{\eta \in \omega^{<\omega}}$ by using a rearrangement map $f : \omega^{<\omega} \to 2^{<\omega}$ and defining $b_\eta := a_{f(\eta)}$. And by applying the modeling properties we obtain an indiscernible tree $\langle c_\eta \rangle_{\eta \in \omega^{<\omega}}$ based on $\langle b_\eta \rangle_{\eta \in \omega^{<\omega}}$. If that indiscernible tree satisfies:

(i) $\{\varphi(x, c_{\eta \restriction n})\}_{n \in \omega}$ is consistent for $\eta \in \omega^\omega$,

(ii) $\{\varphi(x, c_{\eta^\frown n^\frown \nu}), \varphi(x, c_{\eta^\frown m})\}$ for $\eta, \nu \in \omega^{<\omega}$ and $n < m \in \omega$,

then we can obtain an indiscernible witness of SOP_1 by restricting the index set of $\langle c_\eta \rangle_{\eta \in \omega^{<\omega}}$, where $\varphi(x, y)$ is witnessing SOP_1 with $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$. To satisfy (i), the map f should be $\trianglelefteq$-preserving. To satisfy (ii), for ζ, ξ, χ with $\langle \zeta, \xi, \chi \rangle \sim \langle \eta, \eta^\frown n^\frown \nu, \eta^\frown m \rangle$ where $n < m$, f should satisfy $f(\zeta)^\frown 0 \trianglelefteq f(\xi)$ and $f(\zeta)^\frown 1 = f(\chi)$. Hence f should be $<_{lex}$-preserving and $\chi = \zeta^\frown l$ for some $l \in \omega$. But ss- and str-indiscernibility do not guarantee $\chi = \zeta^\frown l$ from $\langle \zeta, \chi \rangle \sim \langle \eta, \eta^\frown m \rangle$. Thus we can not go on.

For the case of s-indiscernibility, fix $n \in \omega$ and define $f : n^{<\omega} \to 2^{<\omega}$ inductively by

$$f(\emptyset) = \emptyset,$$

$$f(\eta^\frown k) = f(\eta)^\frown 0^{(n-k)^\frown} 1,$$

and define $\langle b_\eta \rangle_{\eta \in n^{<\omega}}$ by $b_\eta := a_{f(\eta)}$. Then $\langle b_\eta \rangle_{\eta \in n^{<\omega}}$ satisfies path consistency and for all $\eta, \nu \in n^{<\omega}$ and $m < k < n$, the set $\{\varphi(x, b_{\eta^\frown m^\frown \nu}), \varphi(x, b_{\eta^\frown k})\}$ is inconsistent. By compactness, we have $f : \omega^{<\omega} \to 2^{<\omega}$ such that $\langle b_\eta \rangle_{\eta \in \omega^{<\omega}}$ satisfies path consistency and for all $\eta, \nu \in \omega^{<\omega}$ and $m < k < \omega$, the set $\{\varphi(x, b_{\eta^\frown m^\frown \nu}), \varphi(x, b_{\eta^\frown k})\}$ is inconsistent, where $b_\eta := a_{f(\eta)}$. If we apply the modeling property of s-indiscernibility,

we obtain an s-indiscernible tree $\langle c_\eta \rangle_{\eta \in \omega^{<\omega}}$ s-based on $\langle b_\eta \rangle_{\eta \in \omega^{<\omega}}$. Since the tree language of s-indiscernibility consists unary relation symbols which describe the length of elements of $\omega^{<\omega}$, the inconsistent condition of $\langle b_\eta \rangle_{\eta \in \omega^{<\omega}}$ is inherited to $\langle c_\eta \rangle_{\eta \in \omega^{<\omega}}$. If we restrict $\langle c_\eta \rangle_{\eta \in \omega^{<\omega}}$ to $\langle c_\eta \rangle_{\eta \in 2^{<\omega}}$, we obtain an s-indiscernible witness of SOP_1. But this witness has low homogeneity because the tree language is too precise. For example, in an s-indiscernible tree $\langle a_\eta \rangle_{\eta \in \omega^{<\omega}}$, we can not say that $a_0 a_1 \equiv a_{00} a_{01}$ since $\langle 0, 1 \rangle \not\sim_s \langle 00, 01 \rangle$.

Remark 2.4. This observation is discussing the difficulty of the construction of ss-, str-indiscernible witnesses of SOP_1 from the existence of a witness of SOP_1, does not completely deny its possibility.

2.2 Preserving lemma

From now we find a tree-indiscernibility which preserves witnesses of SOP_1 so that they do not lose their SOP_1 conditions, and does not force the witnesses to be witnesses of SOP_2.

Definition 2.5. For $\bar{\eta} = \langle \eta_0, ..., \eta_n \rangle$, $\bar{\nu} = \langle \nu_0, ..., \nu_n \rangle$ ($\eta_i, \nu_i \in 2^{<\omega}$ for each $i \leq n$), we say $\bar{\eta}$ and $\bar{\nu}$ are α-equivalent ($\bar{\eta} \sim_\alpha \bar{\nu}$) if they satisfy

(i) $\bar{\eta}$ and $\bar{\nu}$ are $\wedge$-closed,

(ii) $\eta_i \trianglelefteq \eta_j$ if and only if $\nu_i \trianglelefteq \nu_j$ for all $i, j \leq n$,

(iii) $\eta_i{}^\frown d \trianglelefteq \eta_j$ if and only if $\nu_i{}^\frown d \trianglelefteq \nu_j$ for all $i, j \leq n$ and $d \leq 1$.

We say $\bar{\eta}$ and $\bar{\nu}$ are β-equivalent ($\bar{\eta} \sim_\beta \bar{\nu}$) if they satisfy (i), (ii), (iii), and

(iv) $\eta_i{}^\frown 1 = \eta_j$ if and only if $\nu_i{}^\frown 1 = \nu_j$ for all $i, j \leq n$.

We say $\bar{\eta}$ and $\bar{\nu}$ are γ-equivalent ($\bar{\eta} \sim_\gamma \bar{\nu}$) if they satisfy (i), (ii), (iii), (iv), and

(v) $\eta_i{}^\frown 0 = \eta_j$ if and only if $\nu_i{}^\frown 0 = \nu_j$ for all $i, j \leq n$.

(vi) $\eta_i = \sigma{}^\frown 0$ for some $\sigma \in 2^{<\omega}$ if and only if $\nu_i = \tau{}^\frown 0$ for some $\tau \in 2^{<\omega}$, for all $i \leq n$.

(vii) $\eta_i = \sigma{}^\frown 1$ for some $\sigma \in 2^{<\omega}$ if and only if $\nu_i = \tau{}^\frown 1$ for some $\tau \in 2^{<\omega}$, for all $i \leq n$.

We say $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ is α-indiscernible (β, γ-indiscernible, resp.) if $\bar{\eta} \sim_\alpha \bar{\nu}$ ($\bar{\eta} \sim_\beta \bar{\nu}$, $\bar{\eta} \sim_\gamma \bar{\nu}$, resp.) implies $a_{\bar{\eta}} \equiv a_{\bar{\nu}}$.

The goal of this chapter is Theorem 2.11, which says that if a theory has a witness of SOP_1, then the theory has a β-indiscernible witness of SOP_1. The outline of proof came from [4, Theorem 2.15]. But the proof of [4] omits some important step, so that the desired conclusion has not been reached completely. We leave complete proof here, and explain what proof of [4] omits, how we complement it.

To show Theorem 2.11 we first prove that if a theory has SOP_1, then it has a γ-indiscernible witness of SOP_1 (Lemma 2.9). And then we obtain a β-indiscernible witness of SOP_1 from this γ-indiscernible one by taking suitable restriction of the parameter part (Theorem 2.11).

Recall the modeling property of α-indiscernibility which appears in [8].

Fact 2.6. [8, Proposition 2.3] *For any $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$, there exists $\langle b_\eta \rangle_{\eta \in 2^{<\omega}}$ such that*

(i) $\langle b_\eta \rangle_{\eta \in 2^{<\omega}}$ is α-indiscernible,

(ii) for any finite set Δ of $\mathcal{L}$-formulas and $\wedge$-closed $\bar{\eta} = \langle \eta_0, ..., \eta_n \rangle$, there exists $\bar{\nu} \sim_\alpha \bar{\eta}$ such that $\bar{b}_{\bar{\eta}} \equiv_\Delta \bar{a}_{\bar{\nu}}$.

In order to make proof shorter we introduce some notations.

Notation 2.7. (i) For each $\eta \in 2^{<\omega}$ with $l(\eta) > 0$, η^- denotes $\eta_{\restriction l(\eta)-1}$ and $t(\eta)$ denotes $\eta(l(\eta) - 1)$.

(ii) For $\bar{\eta} = \langle \eta_0, ..., \eta_n \rangle$, $\mathrm{cl}(\bar{\eta})$ denotes $\langle \eta_0 \wedge \eta_0, ..., \eta_0 \wedge \eta_n \rangle^\frown ... ^\frown \langle \eta_n \wedge \eta_0, ..., \eta_n \wedge \eta_n \rangle$.

Note that $\eta = \eta^-{}^\frown t(\eta)$ for all η with $l(\eta) > 0$. The following lemma will also be useful.

Lemma 2.8. *Suppose $\langle \eta_0, ..., \eta_n \rangle \sim_\gamma \langle \nu_0, ..., \nu_n \rangle$. Then it follows that*

(i) $\eta_i \wedge \eta_j = \eta_k$ if and only if $\nu_i \wedge \nu_j = \nu_k$,

(ii) $\eta_i^- \trianglelefteq \eta_j^-$ if and only if $\nu_i^- \trianglelefteq \nu_j^-$,

(iii) if $\eta_i \perp \eta_j$ then $\eta_i^- \wedge \eta_j^- = \eta_i \wedge \eta_j$,

(iv) $\eta_i{}^\frown d \trianglelefteq \eta_j^-$ if and only if $\nu_i{}^\frown d \trianglelefteq \nu_j^-$,

(v) $\eta_i^-{}^\frown d \trianglelefteq \eta_j^-$ if and only if $\nu_i^-{}^\frown d \trianglelefteq \nu_j^-$,

for all $i, j, k \leq n$ and $d \leq 1$.

Proof. First we show (i). Suppose $\eta_i \wedge \eta_j = \eta_k$. Since $\overline{\eta} = \langle \eta_0, ..., \eta_n \rangle$ and $\overline{\nu} = \langle \nu_0, ..., \nu_n \rangle$ are γ-equivalent, they are closed under $\wedge$. Thus there exists $l \leq n$ such that $\nu_i \wedge \nu_j = \nu_l$. Since $\eta_k \trianglelefteq \eta_i, \eta_j$, we have $\nu_k \trianglelefteq \nu_i, \nu_j$. Thus $\nu_k \trianglelefteq \nu_l$. Since $\nu_l \trianglelefteq \nu_i, \nu_j$, we have $\eta_l \trianglelefteq \eta_i, \eta_j$. Thus $\eta_l \trianglelefteq \eta_k$, and hence $\nu_l \trianglelefteq \nu_k$. Thus $\nu_i \wedge \nu_j = \nu_l = \nu_k$.

To show (ii), suppose $\eta_i^- \trianglelefteq \eta_j^-$ and let $\eta_k = \eta_i \wedge \eta_j$. If $\eta_i \perp \eta_j$, then $\eta_k{}^\frown 0 \trianglelefteq \eta_i$ or $\eta_k{}^\frown 0 \trianglelefteq \eta_j$. Without loss of generality, we may assume $\eta_k{}^\frown 0 \trianglelefteq \eta_i$ so that $\eta_k{}^\frown 1 \trianglelefteq \eta_j$. If $\eta_k{}^\frown 0 \trianglelefteq\!\!\!\!\!\!\cdot\ \ \eta_i$, then we have $\eta_k{}^\frown 0 \trianglelefteq \eta_i^-$, and $\eta_i^- \not\trianglelefteq \eta_j^-$, a contradiction. Thus $\eta_k{}^\frown 0 = \eta_i$. By $\sim_\gamma$, we have $\nu_k{}^\frown 1 \trianglelefteq \nu_j$ and $\nu_k{}^\frown 0 = \nu_i$ which allow us to conclude $\nu_i^- \trianglelefteq \nu_j^-$.

Now suppose that $\eta_i \not\perp \eta_j$ and $\eta_i \trianglelefteq \eta_j$. By $\sim_\gamma$ we have $\nu_i \trianglelefteq \nu_j$ so that $\nu_i^- \trianglelefteq \nu_j^-$. If $\eta_i \not\perp \eta_j$ and $\eta_i \trianglerighteq \eta_j$, then $\eta_i^- \trianglerighteq \eta_j^-$ so that $\eta_i^- = \eta_j^-$. Since we assume $\eta_i \not\perp \eta_j$, we have $\eta_i = \eta_j$. By $\sim_\gamma$ we conclude $\nu_i = \nu_j$ and hence $\nu_i^- = \nu_j^-$. Therefore $\nu_i^- \trianglelefteq \nu_j^-$.

To show (iii), suppose $\eta_i \perp \eta_j$. Let $\eta_k = \eta_i \wedge \eta_j$. As above, we may assume $\eta_k{}^\frown 0 \trianglelefteq \eta_i$ and $\eta_k{}^\frown 1 \trianglelefteq \eta_j$. Thus $\eta_k = \eta_i^-$ or $\eta_k{}^\frown 0 \trianglelefteq \eta_i^-$, and $\eta_k = \eta_j^-$ or $\eta_k{}^\frown 1 \trianglelefteq \eta_j$. In any case, it is easy to verify that $\eta_k = \eta_i^- \wedge \eta_j^-$. Thus we have $\eta_i^- \wedge \eta_j^- = \eta_i \wedge \eta_j$.

To show (iv) we suppose $\eta_i{}^\frown d \trianglelefteq \eta_j^-$ for some $d \leq 1$. Then we have $\eta_i{}^\frown d \trianglelefteq \eta_j$ and $\eta_i{}^\frown d \neq \eta_j$. By $\sim_\gamma$, we have $\nu_i{}^\frown d \trianglelefteq \nu_j$ and $\nu_i{}^\frown d \neq \nu_j$. Thus $\nu_i{}^\frown d \trianglelefteq \nu_j^-$.

For (v), suppose $\eta_i^-{}^\frown d \trianglelefteq \eta_j^-$ for some $d \leq 1$. First we assume $t(\eta_i) = d$. Then $\eta_i = \eta_i^-{}^\frown d \trianglelefteq \eta_j^- \trianglelefteq\!\!\!\!\!\!\cdot\ \ \eta_j$. By $\sim_\gamma$, we have $t(\nu_i) = d$ and $\nu_i \trianglelefteq\!\!\!\!\!\!\cdot\ \ \nu_j$, so that $\nu_i^-{}^\frown d = \nu_i \trianglelefteq \nu_j^-$. Now we assume $t(\eta_i) \neq d$. Then $\eta_i = \eta_i^-{}^\frown (d-1)^2$ and $\eta_j \trianglerighteq\!\!\!\!\!\!\cdot\ \ \eta_j^- \trianglerighteq \eta_i{}^\frown d$. Thus $\eta_i \perp \eta_j$ and $\eta_k = \eta_i \wedge \eta_j = \eta_i^-$ for some $k \leq n$. Since $\eta_k{}^\frown (d-1)^2 = \eta_i$ and $\eta_k{}^\frown d \trianglelefteq\!\!\!\!\!\!\cdot\ \ \eta_j$, we have $\nu_k{}^\frown (d-1)^2 = \nu_i$ and $\nu_k{}^\frown d \trianglelefteq\!\!\!\!\!\!\cdot\ \ \nu_j$ by $\sim_\gamma$. Thus we have $\nu_i^-{}^\frown d = \nu_k{}^\frown d \trianglelefteq\!\!\!\!\!\!\cdot\ \ \nu_j$ so that $\nu_i^-{}^\frown d \trianglelefteq \nu_j^-$, as desired. □

Lemma 2.9. *Suppose $\varphi(x, \bar{y})$ witnesses SOP$_1$. Then there exists a γ-indiscernible tree $\langle d_\eta \rangle_{\eta \in 2^{<\omega}}$ which witnesses SOP$_1$ with φ.*

Proof. Suppose $\varphi(x, \bar{y})$ witnesses SOP$_1$ with $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$. For each $\eta \in 2^{<\omega}$, put $b_\eta = a_{\eta^\frown 0} a_{\eta^\frown 1}$. By Fact 2.6, there exists an α-indiscernible $\langle c_\eta \rangle_{\eta \in 2^{<\omega}}$ such that for any $\bar{\eta}$ and finite subset Δ of $\mathcal{L}$-formulas, there exists $\bar{\nu}$ such that $\bar{\nu} \sim_\alpha \bar{\eta}$ and $\bar{b}_{\bar{\nu}} \equiv_\Delta \bar{c}_{\bar{\eta}}$. Note

that c_η is of the form $c^0_\eta c^1_\eta$ where $|c^0_\eta| = |c^1_\eta| = |\bar{y}|$ for each $\eta \in 2^{<\omega}$. For each $\eta \in 2^{<\omega}$ with $l(\eta) \geq 1$, we define d'_η by

$$d'_\eta = \begin{cases} c^0_{\eta^-} & \text{if } \ t(\eta) = 0 \\ c^1_{\eta^-} & \text{if } \ t(\eta) = 1 \end{cases}$$

and put $d_\eta = d'_{0^\frown\eta}$ for each $\eta \in 2^{<\omega}$. We show that φ witnesses SOP_1 with $\langle d_\eta \rangle_{\eta \in 2^{<\omega}}$ and $\langle d_\eta \rangle_{\eta \in 2^{<\omega}}$ is γ-indiscernible.

Claim 1. For all $\eta, \nu \in 2^{<\omega}$, $\{\varphi(x, d_{\eta^\frown 1}), \varphi(x, d_{\eta^\frown 0^\frown \nu})\}$ is inconsistent.

Proof. Define ψ_0 and ψ_1 by

$$\psi_0(x; \bar{y_0}, \bar{y_1}) = \varphi(x, \bar{y_0}) \wedge \bar{y_1} = \bar{y_1},$$
$$\psi_1(x; \bar{y_0}, \bar{y_1}) = \varphi(x, \bar{y_1}) \wedge \bar{y_0} = \bar{y_0}.$$

Suppose $\nu = \langle\rangle$. Clearly $\langle 0^\frown\eta\rangle$ is $\wedge$-closed because the tuple has only one component. Hence there exists $\mu \in 2^{<\omega}$ such that $\mu \sim_\alpha 0^\frown\eta$ and $b_\mu \equiv_\Delta c_{0^\frown\eta}$, where $\Delta = \{\psi_0, \psi_1\}$. Since $\{\varphi(x, a_{\mu^\frown 1}), \varphi(x, a_{\mu^\frown 0})\}$ is inconsistent, it follows that $\{\psi_1(x, b_\mu), \psi_0(x, b_\mu)\}$ is inconsistent, $\{\psi_1(x, c_{0^\frown\eta}), \psi_0(x, c_{0^\frown\eta})\}$ is inconsistent, $\{\varphi(x, c^1_{0^\frown\eta}), \varphi(x, c^0_{0^\frown\eta})\}$ is inconsistent, $\{\varphi(x, d'_{0^\frown\eta^\frown 1}), \varphi(x, d'_{0^\frown\eta^\frown 0})\}$ is inconsistent, and hence $\{\varphi(x, d_{\eta^\frown 1}), \varphi(x, d_{\eta^\frown 0^\frown\nu})\}$ is inconsistent.

Now we suppose $\nu \neq \langle\rangle$. Since $\langle 0^\frown\eta, 0^\frown\eta^\frown 0^\frown\nu^-\rangle$ is $\wedge$-closed, there exists $\mu, \lambda \in 2^{<\omega}$ such that

$$\langle \mu, \lambda\rangle \sim_\alpha \langle 0^\frown\eta, 0^\frown\eta^\frown 0^\frown\nu^-\rangle,$$
$$b_\mu b_\lambda \equiv_\Delta c_{0^\frown\eta} c_{0^\frown\eta^\frown 0^\frown\nu^-}.$$

Then $\mu^\frown 0 \trianglelefteq \lambda \trianglelefteq \lambda^\frown t(\nu)$ by $\sim_\alpha$. Thus $\{\varphi(x, a_{\mu^\frown 1}), \varphi(x, a_{\lambda^\frown t(\nu)})\}$ is inconsistent. Thus $\{\psi_1(x, b_\mu), \psi_{t(\nu)}(x, b_\lambda)\}$ is inconsistent, $\{\psi_1(x, c_{0^\frown\eta}), \psi_{t(\nu)}(x, c_{0^\frown\eta^\frown 0^\frown\nu^-})\}$ is inconsistent, and hence $\{\varphi(x, d_{\eta^\frown 1}), \varphi(x, d_{\eta^\frown 0^\frown\nu})\}$ is inconsistent. □

Claim 2. For all $\eta \in 2^{<\omega}$, $\{\varphi(x, d_{\eta\restriction n}) : n \in \omega\}$ is consistent.

Proof. Since $d_{\langle\rangle} = c^0_{\langle\rangle}$ and $d_{\eta\restriction(n+1)} = c^{\eta(n)}_{0^\frown(\eta\restriction n)}$ for all $n \in \omega$, it is enough to show that $\{\psi_0(x, c_{\langle\rangle})\} \cup \{\psi_{\eta(n)}(x, c_{0^\frown(\eta\restriction n)}) : n \in \omega\}$ is consistent. We use compactness. Choose any finite subset of $\{\psi_0(x, c_{\langle\rangle})\} \cup \{\psi_{\eta(n)}(x, c_{0^\frown(\eta\restriction n)}) : n \in \omega\}$. We may assume that this subset is of the form

$$\{\psi_0(x, c_{\langle\rangle}), \psi_{\eta(n_1)}(x, c_{0^\frown(\eta\restriction n_1)}), ..., \psi_{\eta(n_m)}(x, c_{0^\frown(\eta\restriction n_m)})\}$$

for some $m \in \omega$ and $n_1 < ... < n_m < \omega$. Clearly $\langle\emptyset, 0^\frown(\eta \restriction n_1), ..., 0^\frown(\eta \restriction n_m)\rangle$ is $\wedge$-closed. Thus there exists $\nu_0, \nu_1, ..., \nu_m \in 2^{<\omega}$ such that

$$\langle \nu_0, \nu_1, ..., \nu_m\rangle \sim_\alpha \langle\langle\rangle, 0^\frown(\eta \restriction n_1), ..., 0^\frown(\eta \restriction n_m)\rangle$$
$$b_{\nu_0} b_{\nu_1} ... b_{\nu_m} \equiv_\Delta c_{\langle\rangle} c_{0^\frown(\eta\restriction n_1)} ... c_{0^\frown(\eta\restriction n_m)},$$

where $\Delta = \{\psi_0, \psi_1\}$. Since

$$\langle\rangle^\frown 0 \trianglelefteq 0^\frown(\eta \restriction n_1)^\frown\eta(n_1) \trianglelefteq ... \trianglelefteq 0^\frown(\eta \restriction n_m)^\frown\eta(n_m),$$

we have

$$\nu_0{}^\frown 0 \trianglelefteq \nu_1{}^\frown\eta(n_1) \trianglelefteq ... \trianglelefteq \nu_m{}^\frown\eta(n_m)$$

by $\sim_\alpha$. Thus

$$\{\varphi(x, a_{\nu_0{}^\frown 0}), \varphi(x, a_{\nu_1{}^\frown\eta(n_1)}), ..., \varphi(x, a_{\nu_m{}^\frown\eta(n_m)})\}$$

is consistent and hence

$$\{\psi_0(x, b_{\nu_0}), \psi_{\eta(n_1)}(x, b_{\nu_1}), ..., \psi_{\eta(n_m)}(x, b_{\nu_m})\}$$

is consistent. Since $b_{\nu_0} b_{\nu_1} ... b_{\nu_m} \equiv_\Delta c_{\langle\rangle} c_{0^\frown(\eta\restriction n_1)} ... c_{0^\frown(\eta\restriction n_m)}$,

$$\{\psi_0(x, c_{\langle\rangle}), \psi_{\eta(n_1)}(x, c_{0^\frown(\eta\restriction n_1)}), ..., \psi_{\eta(n_m)}(x, c_{0^\frown(\eta\restriction n_m)})\}$$

is also consistent. By compactness, $\{\psi_0(x, c_{\langle\rangle})\} \cup \{\psi_{\eta(n)}(x, c_{0^\frown(\eta\restriction n)}) : n \in \omega\}$ is consistent. □

Thus φ witnesses SOP_1 with $\langle d_\eta \rangle_{\eta \in 2^{<\omega}}$.

Now we prove that $\langle d_\eta \rangle_{\eta \in 2^{<\omega}}$ is γ-indiscernible. Suppose that $\langle \eta_0, ..., \eta_n \rangle \sim_\gamma \langle \nu_0, ..., \nu_n \rangle$. For each $i \leq n$, let $\sigma_i = 0^\frown \eta_i$ and $\tau_i = 0^\frown \nu_i$. It is enough to show that

$$d'_{\sigma_0} ... d'_{\sigma_n} \equiv d'_{\tau_0} ... d'_{\tau_n}$$

by the choice of $\langle d_\eta \rangle_{\eta \in 2^{<\omega}}$. First, we show $\langle \sigma_0^-, ..., \sigma_n^- \rangle \sim_\alpha \langle \tau_0^-, ..., \tau_n^- \rangle$. Note that $\langle \sigma_0^-, ..., \sigma_n^- \rangle$ and $\langle \tau_0^-, ..., \tau_n^- \rangle$ may not be $\wedge$-closed. Thus we prove $\mathrm{cl}(\langle \sigma_0^-, ..., \sigma_n^- \rangle) \sim_\alpha \mathrm{cl}(\langle \tau_0^-, ..., \tau_n^- \rangle)$. Note also that $\langle \sigma_0, ..., \sigma_n \rangle$ and $\langle \tau_0, ..., .\tau_n \rangle$ are γ-equivalent and still $\wedge$-closed. It allows us to use Lemma 2.8. We will use Lemma 2.8 frequently without much mention.

Claim 3. $\mathrm{cl}(\langle \sigma_0^-, ..., \sigma_n^- \rangle) \sim_\alpha \mathrm{cl}(\langle \tau_0^-, ..., \tau_n^- \rangle)$.

Proof. **Subclaim 1.** $\sigma_i^- \wedge \sigma_j^- \trianglelefteq \sigma_k^- \wedge \sigma_l^-$ if and only if $\tau_i^- \wedge \tau_j^- \trianglelefteq \tau_k^- \wedge \tau_l^-$ for all $i, j, k, l \leq n$.

Proof. Suppose $\sigma_i^- \wedge \sigma_j^- \trianglelefteq \sigma_k^- \wedge \sigma_l^-$. It is enough to show that $\tau_i^- \wedge \tau_j^- \trianglelefteq \tau_k^- \wedge \tau_l^-$.

Case 1 [$\sigma_i \perp \sigma_j$]. Then $\tau_i \perp \tau_j$. By Lemma 2.8 we have $\sigma_i \wedge \sigma_j = \sigma_i^- \wedge \sigma_j^- \trianglelefteq \sigma_k^- \wedge \sigma_l^- \trianglelefteq \sigma_k, \sigma_l$ and similarly $\tau_i \wedge \tau_j \trianglelefteq \tau_k, \tau_l$. If $\tau_i \wedge \tau_j = \tau_k$, then $\sigma_i \wedge \sigma_j = \sigma_k$ and hence $\sigma_k^- \wedge \sigma_l^- \trianglelefteq \sigma_k^- \ntrianglelefteq \sigma_i \wedge \sigma_j = \sigma_i^- \wedge \sigma_j^-$, it is a contradiction. Thus $\tau_i \wedge \tau_j \ntrianglelefteq \tau_k, \tau_l$ and hence $\tau_i \wedge \tau_j \trianglelefteq \tau_k^-, \tau_l^-$. Therefore we have $\tau_i^- \wedge \tau_j^- = \tau_i \wedge \tau_j \trianglelefteq \tau_k^- \wedge \tau_l^-$.

Case 2 [$\sigma_i \not\perp \sigma_j$]. We may assume $\sigma_i \trianglelefteq \sigma_j$. Then $\sigma_i^- \trianglelefteq \sigma_j^-$. Thus $\sigma_i^- = \sigma_i^- \wedge \sigma_j^- \trianglelefteq \sigma_k^- \wedge \sigma_l^- \trianglelefteq \sigma_k^-, \sigma_l^-$. By Lemma 2.8(ii) and $\sim_\gamma$, we have $\tau_i^- \wedge \tau_j^- = \tau_i^- \trianglelefteq \tau_k^-, \tau_l^-$. Thus $\tau_i^- \wedge \tau_j^- \trianglelefteq \tau_k^- \wedge \tau_l^-$. This proves Subclaim 1. □

Subclaim 2. $(\sigma_i^- \wedge \sigma_j^-)^\frown d \trianglelefteq \sigma_k^- \wedge \sigma_l^-$ if and only if $(\tau_i^- \wedge \tau_j^-)^\frown d \trianglelefteq \tau_k^- \wedge \tau_l^-$, for all $i, j, k, l \leq n$ and $d \leq 1$.

Proof. Suppose $(\sigma_i^- \wedge \sigma_j^-)^\frown \langle d \rangle \trianglelefteq \sigma_k^- \wedge \sigma_l^-$. Without loss of generality, we may assume $d = 0$ and it suffices to show that $(\tau_i^- \wedge \tau_j^-)^\frown 0 \trianglelefteq \tau_k^- \wedge \tau_l^-$.

Case 1 [$\sigma_i \perp \sigma_j$]. Then $\sigma_i^- \wedge \sigma_j^- = \sigma_i \wedge \sigma_j$ and $\tau_i^- \wedge \tau_j^- = \tau_i \wedge \tau_j$. Since $\langle \sigma_0, ..., \sigma_n \rangle$ is $\wedge$-closed, there exists $m \leq n$ such that $\sigma_i \wedge \sigma_j = \sigma_m$ and by Lemma 2.8(i), $\tau_i \wedge \tau_j = \tau_m$. Since $\sigma_m{}^\frown 0 = (\sigma_i \wedge \sigma_j)^\frown 0 = (\sigma_i^- \wedge \sigma_j^-)^\frown 0 \trianglelefteq \sigma_k^- \wedge \sigma_l^- \trianglelefteq \sigma_k^-, \sigma_l^-$, it follows that $\tau_m{}^\frown 0 \trianglelefteq \tau_k^-, \tau_l^-$ by Lemma 2.8(iv). Since $\tau_i^- \wedge \tau_j^- = \tau_i \wedge \tau_j = \tau_m$, we have $(\tau_i^- \wedge \tau_j^-)^\frown 0 \trianglelefteq \tau_k^- \wedge \tau_l^-$.

Case 2 [$\sigma_i \not\perp \sigma_j$]. We may assume $\sigma_i \trianglelefteq \sigma_j$. Then $\sigma_i^- \trianglelefteq \sigma_j^-$ and hence $\sigma_i^-{}^\frown 0 = (\sigma_i^- \wedge \sigma_j^-)^\frown 0 \trianglelefteq \sigma_k^- \wedge \sigma_l^- \trianglelefteq \sigma_k^-, \sigma_l^-$. Thus $\tau_i^-{}^\frown 0 \trianglelefteq \tau_k^-, \tau_l^-$ by Lemma 2.8(v). Since $\tau_i \trianglelefteq \tau_j$ by $\sim_\gamma$, we have $\tau_i^- \trianglelefteq \tau_j^-$ and hence $(\tau_i^- \wedge \tau_j^-)^\frown 0 = \tau_i^-{}^\frown 0 \trianglelefteq \tau_k^- \wedge \tau_l^-$. This proves Subclaim 2. □

Clearly $\mathrm{cl}(\langle \sigma_0^-, ..., \sigma_n^- \rangle)$ and $\mathrm{cl}(\langle \tau_0^-, ..., \tau_n^- \rangle)$ are $\wedge$-closed. Together with Subclaim 1 and 2, it proves Claim 3. □

By α-indiscernibility of $\langle c_\eta \rangle_{\eta \in 2^{<\omega}}$, we have $\bar{c}_{\mathrm{cl}(\langle \sigma_0^-, ..., \sigma_n^- \rangle)} \equiv \bar{c}_{\mathrm{cl}(\langle \tau_0^-, ..., \tau_n^- \rangle)}$. In particular, we have $c_{\sigma_0^-} ... c_{\sigma_n^-} \equiv c_{\tau_0^-} ... c_{\tau_n^-}$. By definition of $\langle d'_\eta \rangle_{\eta \in 2^{<\omega}}$,

$$d'_{\sigma_0^-{}^\frown 0} d'_{\sigma_0^-{}^\frown 1} ... d'_{\sigma_n^-{}^\frown 0} d'_{\sigma_n^-{}^\frown 1} \equiv d'_{\tau_0^-{}^\frown 0} d'_{\tau_0^-{}^\frown 1} ... d'_{\tau_n^-{}^\frown 0} d'_{\tau_n^-{}^\frown 1}.$$

For each $i \leq n$ and $d \leq 1$, let $\xi_{2i+d} = \sigma_i^-{}^\frown \langle d \rangle$ and $\zeta_{2i+d} = \tau_i^-{}^\frown \langle d \rangle$. Then we have

$$d'_{\xi_0} ... d'_{\xi_{2n+1}} \equiv d'_{\zeta_0} ... d'_{\zeta_{2n+1}}.$$

Note that in general, if $m_{\xi_0}...m_{\xi_k} \equiv m_{\zeta_0}...m_{\zeta_k}$ and $i_0 < ... < i_e \leq k$, then $m_{\xi_{i_0}}...m_{\xi_{i_e}} \equiv m_{\zeta_{i_0}}...m_{\zeta_{i_e}}$. Since $\langle \sigma_0, ..., \sigma_n \rangle \sim_\gamma \langle \tau_0, ..., \tau_n \rangle$, we have $2i + t(\sigma_i) = 2i + t(\tau_i)$ for all $i \leq n$. Thus

$$\begin{aligned} d'_{\sigma_0}...d'_{\sigma_n} = d'_{\sigma_0^- \frown t(\sigma_0)}...d'_{\sigma_n^- \frown t(\sigma_n)} = d'_{\xi_{t(\sigma_0)}}...d'_{\xi_{2n+t(\sigma_n)}} &\equiv d'_{\zeta_{t(\tau_0)}}...d'_{\zeta_{2n+t(\tau_n)}} \\ &= d'_{\tau_0^- \frown t(\tau_0)}...d'_{\tau_n^- \frown t(\tau_n)} \\ &= d'_{\tau_0}...d'_{\tau_n} \end{aligned}$$

as desired. This shows that $\langle d_\eta \rangle_{\eta \in 2^{<\omega}}$ is γ-indiscernible, and completes proof of Lemma 2.9. □

Note that even if $i_0 < ... < i_e \leq k$, $j_0 < ... < j_e \leq k$ and $m_{\xi_0}...m_{\xi_k} \equiv m_{\zeta_0}...m_{\zeta_k}$, it is not sure that $m_{\xi_{i_0}}...m_{\xi_{i_e}} \equiv m_{\zeta_{j_0}}...m_{\zeta_{j_e}}$. Hence if we want to say $d'_{\sigma_0}...d'_{\sigma_n} \equiv d'_{\tau_0}...d'_{\tau_n}$ from

$$d'_{\sigma_0^- \frown 0} d'_{\sigma_0^- \frown 1} ... d'_{\sigma_n^- \frown 0} d'_{\sigma_n^- \frown 1} \equiv d'_{\tau_0^- \frown 0} d'_{\tau_0^- \frown 1} ... d'_{\tau_n^- \frown 0} d'_{\tau_n^- \frown 1}$$

in the last paragraph of proof of Lemma 2.9, it must be guaranteed that $t(\sigma_i) = t(\tau_i)$ for each $i \leq n$. This is why we introduce $\sim_\gamma$ and find a γ-indiscernible witness of SOP_1 first, not directly find β-indiscernible one as in [4]. The proof in [4] tries to show directly (without using γ-indiscernibility) the existence of a β-indiscernible witness of SOP_1, hence it ends incompletely due to the aforementioned problem.

We will need the following lemma when we prove Theorem 2.11.

Lemma 2.10. *If $h : 2^{<\omega} \to 2^{<\omega}$ satisfies $h(\eta)^\frown d \trianglelefteq h(\eta^\frown d)$ for all $\eta \in 2^{<\omega}$ and $d \leq 1$, then they satisfy*

(i) $h(\eta) \trianglelefteq h(\nu)$ if and only if $\eta \trianglelefteq \nu$,

(ii) $h(\eta) \wedge h(\nu) = h(\eta \wedge \nu)$

for all $\eta, \nu \in 2^{<\omega}$.

Proof. To show (i), first we prove that if $\eta \trianglelefteq \nu$ then $h(\eta) \trianglelefteq h(\nu)$. Note that if $\eta \trianglelefteq \nu$, then there exists $d \leq 1$ and σ such that $\nu = \eta^\frown d^\frown \sigma$. Thus we show that $h(\eta) \trianglelefteq h(\eta^\frown d^\frown \sigma)$ for all $\eta, \sigma \in 2^{<\omega}$ and $d \leq 1$, by using induction on σ. If $\sigma = \emptyset$, then $h(\eta) \trianglelefteq h(\eta)^\frown d \trianglelefteq h(\eta^\frown d) = h(\eta^\frown d^\frown \sigma)$. Suppose $\sigma \neq \emptyset$ and $h(\eta) \trianglelefteq h(\eta^\frown d^\frown \sigma')$ for all $\sigma' \trianglelefteq \sigma$. Then we have $h(\eta) \trianglelefteq h(\eta^\frown d^\frown \sigma^-) \trianglelefteq h(\eta^\frown d^\frown \sigma^-)^\frown t(\sigma) \trianglelefteq h(\eta^\frown d^\frown \sigma^-{}^\frown t(\sigma)) = h(\eta^\frown d^\frown \sigma)$. Thus we have $h(\eta) \trianglelefteq h(\nu)$ whenever $\eta \trianglelefteq \nu$ and hence we have $h(\eta) \trianglelefteq h(\nu)$ whenever $\eta \trianglelefteq \nu$.

To show the converse, let us suppose $\eta \not\trianglelefteq \nu$. Then $\nu \trianglelefteq \eta$ or $\eta \perp \nu$. If $\nu \trianglelefteq \eta$, then as we observed, $h(\nu) \trianglelefteq h(\eta)$. Thus $h(\eta) \not\trianglelefteq h(\nu)$. If $\eta \perp \nu$, then $\zeta^\frown d \trianglelefteq \eta$ and $\zeta^\frown (d-1)^2 \trianglelefteq \nu$ for some $d \leq 1$ where $\zeta = \eta \wedge \nu$. We may assume $d = 0$. Thus we have

$$h(\zeta) \trianglelefteq h(\zeta)^\frown 0 \trianglelefteq h(\zeta^\frown 0) \trianglelefteq h(\eta) \text{ and}$$
$$h(\zeta) \trianglelefteq h(\zeta)^\frown 1 \trianglelefteq h(\zeta^\frown 1) \trianglelefteq h(\nu)$$

and this shows $h(\eta) \perp h(\nu)$ which implies $h(\eta) \not\trianglelefteq h(\nu)$.

Now we show (ii). If $\eta \trianglelefteq \nu$, then $h(\eta) \trianglelefteq h(\nu)$ by (i) and hence we have $h(\eta \wedge \nu) = h(\eta) = h(\eta) \wedge h(\nu)$. Thus we assume $\eta \perp \nu$. We may assume $\zeta^\frown 0 \trianglelefteq \eta$ and $\zeta^\frown 1 \trianglelefteq \nu$ where $\zeta = \eta \wedge \nu$. As we observed above, $h(\zeta)^\frown 0 \trianglelefteq h(\eta)$ and $h(\zeta)^\frown 1 \trianglelefteq h(\nu)$. Therefore $h(\eta) \wedge h(\nu) = h(\zeta) = h(\eta \wedge \nu)$. □

Theorem 2.11. *If* $\varphi(x, y)$ *witnesses SOP*$_1$*, then there exists a* β*-indiscernible tree* $\langle e_\eta \rangle_{\eta \in 2^{<\omega}}$ *which witnesses SOP*$_1$ *with* φ*.*

Proof. By Lemma 2.9, there exists a γ-indiscernible tree $\langle d_\eta \rangle_{\eta \in 2^{<\omega}}$ which witnesses

SOP$_1$ with φ. Define a map $h : 2^{<\omega} \to 2^{<\omega}$ by

$$h(\eta) = \begin{cases} \langle\rangle & \text{if } \eta = \langle\rangle \\ h(\eta^-)^\frown 01 & \text{if } t(\eta) = 0 \\ h(\eta^-)^\frown 1 & \text{if } t(\eta) = 1, \end{cases}$$

and put $e_\eta = d_{h(\eta)}$ for each $\eta \in 2^{<\omega}$.

Claim 1. $\langle e_\eta \rangle_{\eta \in 2^{<\omega}}$ is β-indiscernible.

Proof. Let $\langle \eta_0, ..., \eta_n \rangle \sim_\beta \langle \nu_0, ..., \nu_n \rangle$. It is enough to show that $h(\overline{\eta}) = \langle h(\eta_0), ..., h(\eta_n) \rangle$ $\sim_\gamma \langle h(\nu_0), ..., h(\nu_n) \rangle = h(\overline{\nu})$. We recall Definition 2.5. Note that h satisfies the assumption of Lemma 2.10. Thus $h(\bar{\eta})$ and $h(\bar{\nu})$ satisfy (i), (ii) of Definition 2.5. To show $h(\bar{\eta})$ and $h(\bar{\nu})$ satisfy (iii) we suppose $h(\eta_i)^\frown d \trianglelefteq h(\eta_j)$ for some $d \leq 1$. Then $h(\eta_i) \trianglelefteq h(\eta_j)$ and hence $\eta_i \trianglelefteq \eta_j$ by Lemma 2.10. Thus either $\eta_i{}^\frown d \trianglelefteq \eta_j$ or $\eta_i{}^\frown (d-1)^2 \trianglelefteq \eta_j$. If $\eta_i{}^\frown (d-1)^2 \trianglelefteq \eta_j$, then we have

$$h(\eta_i)^\frown (d-1)^2 \trianglelefteq h(\eta_i{}^\frown (d-1)^2) \trianglelefteq h(\eta_j),$$

a contradiction. Thus $\eta_i{}^\frown d \trianglelefteq \eta_j$. Since $\bar{\eta} \sim_\beta \bar{\nu}$, we have $\nu_i{}^\frown d \trianglelefteq \nu_j$. Hence

$$h(\nu_i)^\frown d \trianglelefteq h(\nu_i{}^\frown d) \trianglelefteq h(\nu_j),$$

and therefore $h(\bar{\eta})$ and $h(\bar{\nu})$ satisfy (iii). Now we suppose $h(\eta_i)^\frown 1 = h(\eta_j)$. Then we have $h(\eta_i{}^\frown 1) = h(\eta_i)^\frown 1 = h(\eta_j)$ by definition of h. Since h is an injection, we have $\eta_i{}^\frown 1 = \eta_j$. Since $\bar{\eta} \sim_\beta \bar{\nu}$, we have $\nu_i{}^\frown 1 = \nu_j$. Therefore $h(\nu_i)^\frown 1 = h(\nu_i{}^\frown 1) = h(\nu_j)$. This shows that $h(\bar{\eta})$ and $h(\bar{\nu})$ satisfy (iv).

By definition of h, there is no η such that $t(h(\eta)) = 0$. Thus (v) is vacuously true. By the same reason, $h(\bar{\eta})$, $h(\bar{\nu})$ satisfy (vi) and (vii). □

Claim 2. φ witnesses SOP$_1$ with $\langle e_\eta \rangle_{\eta \in 2^{<\omega}}$.

Proof. Choose any $\eta \in 2^\omega$. Then $\{h(\eta \restriction n) : n \in \omega\}$ is linearly ordered by $\trianglelefteq$ by Lemma 2.10. Thus $\{\varphi(x, e_{\eta\restriction n}) : n \in \omega\} = \{\varphi(x, d_{h(\eta\restriction n)}) : n \in \omega\}$ is consistent. Choose any $\eta, \nu \in 2^{<\omega}$. Then

$$\{\varphi(x, e_{\eta^\frown 0^\frown \nu}), \varphi(x, e_{\eta^\frown 1})\} = \{\varphi(x, d_{h(\eta^\frown 0^\frown \nu)}), \varphi(x, d_{h(\eta^\frown 1)})\}$$

and $h(\eta)^\frown 0 \trianglelefteq h(\eta^\frown 0) \trianglelefteq h(\eta^\frown 0^\frown \nu)$ and $h(\eta^\frown 1) = h(\eta)^\frown 1$. Therefore the set $\{\varphi(x, e_{\eta^\frown 0^\frown \nu}), \varphi(x, e_{\eta^\frown 1})\}$ is inconsistent. □

This completes proof of Theorem 2.11. □

Thus if there exists a witness φ of SOP_1, then we may choose a β-indiscernible tree of parameters which witnesses SOP_1 with φ.

Chapter 3

Antichain tree property

3.1 Relationship between ATP and other tree properties

In this chapter, we introduce a notion of tree property which is called *antichain tree property (ATP)* and explain how to construct an antichain tree in a SOP_1-NSOP_2 theory. In short, the concept of ATP is the opposite of the concept of SOP_2 in the following sense.

Definition 3.1. (i) A subset X of $2^{<\omega}$ is called an *antichain* if $\eta \perp \nu$ for all $\eta, \nu \in X$.

(ii) A tuple $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ is called an *antichain tree* if for every $X \subseteq 2^{<\omega}$, $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent if and only if X is an antichain in $2^{<\omega}$.

(iii) We say φ has *antichain tree property (ATP)* if φ forms an antichain tree with some $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$. We say T *has ATP* or T *is ATP* if there is a formula which has ATP modulo T. We say T *is NATP* if T does not have ATP.

In other words, a subset X of $2^{<\omega}$ is an antichain if and only if X is pairwisely incomparable. Therefore, we can write Definition 3.1 again as

Definition 3.2. $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ is called an antichain tree if for all $X \subseteq 2^{<\omega}$, $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent if and only if X is pairwisely 'incomparable'.

And we can write the definition SOP_2 again as

Definition 3.3. We say $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ witnesses SOP_2 if for all $X \subseteq 2^{<\omega}$, $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent if and only if X is pairwisely 'comparable'.

In this sense, ATP can be thought of as having the opposite characteristics to SOP_2.

If an antichain tree $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ is given, we can find a witness of SOP_1 and a witness of TP_2 by restricting the parameter part $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ as follows.

Proposition 3.4. *If $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ is an antichain tree, then $\varphi(x,y)$ witnesses SOP$_1$.*

Proof. By compantnss, it is enough to show that for each $n \in \omega$, there exists $h_n : 2^{\leq n} \to 2^{<\omega}$ such that

(i) $\{\varphi(x, b_{\eta \restriction i}) : i \leq n\}$ is consistent for all $\eta \in 2^n$,

(ii) $\{\varphi(x, b_{\eta^\frown 0^\frown \nu}), \varphi(x, b_{\eta^\frown 1})\}$ is inconsistent for all $\eta, \nu \in 2^{<n}$ with $\eta \,^\frown 0 \,^\frown \nu$, $\eta \,^\frown 1 \in 2^{\leq n}$,

where $b_\eta = a_{h_n(\eta)}$ for each $\eta \in 2^{\leq n}$. We use induction on $n \in \omega$. Define $h_0 : 2^{\leq 0} \to 2^{<\omega}$ by $h_0(\emptyset) = 1$. For $n \in \omega$, assume such h_n exists. Define $h_{n+1} : 2^{\leq n+1} \to 2^{<\omega}$ by

$$h_{n+1}(\eta) = \begin{cases} 1 & \text{if } \eta = \langle\rangle \\ \langle 011 \rangle^\frown h_n(\nu) & \text{if } \eta = 0^\frown \nu \text{ for some } \nu \in 2^{\leq n} \\ \langle 0 \rangle^\frown h_n(\nu) & \text{if } \eta = 1^\frown \nu \text{ for some } \nu \in 2^{\leq n}. \end{cases}$$

It is easy to show that $\langle \varphi(x,y), \langle b_\eta \rangle_{\eta \in 2^{\leq n}} \rangle$ satisfies (i) and (ii) for each $n \in \omega$ where $b_\eta = a_{h_n(\eta)}$. □

Proposition 3.5. *If $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ is an antichain tree, then $\varphi(x,y)$ witnesses TP_2.*

Proof. By compactness, it is enough to show that for any $n \in \omega$, there exists $\langle b_{i,j} \rangle_{i,j<n}$ such that $\{\varphi(x, b_{i,j}) : j < n\}$ is 2-inconsistent for each $i < n$, and $\{\varphi(x, b_{i,f(i)}) : i < n\}$ is consistent for all $f : n \to n$. Fix $n \in \omega$. Choose any antichain $\{\eta_0, ..., \eta_{n-1}\}$ in $2^{<\omega}$. Define $h_n : n \times n \to 2^{<\omega}$ by

$$h_n(i,j) = \eta_i{}^\frown \langle 0^j \rangle.$$

Put $b_{i,j} = a_{h_n(i,j)}$ for each $i, j < n$. Then $\{\varphi(x, b_{i,j}) : j < n\}$ is 2-inconsistent for all $i < n$ and $\{\varphi(x, b_{i,f(i)}) : i < n\}$ is consistent for all $f : n \to n$. □

Hence if a theory has an antichain tree, then the theory is TP_2 and SOP_1.

3.2 Construction of an antichain tree on an SOP_1-NSOP_2 formula

From now on we will construct an antichain tree under some classification theoretical hypothesis. We begin the construction with the following lemmas.

Lemma 3.6. *For any $c : 2^{<\omega_1} \to \omega$, one can find $g : 2^{<\omega} \to 2^{<\omega_1}$ and $i_0 \in \omega$ such that*

(i) $g(\eta)^\frown l \trianglelefteq g(\eta^\frown l)$ *for all* $\eta \in 2^{<\omega}$ *and* $l \leq 1$,

(ii) $c(g(\eta)) = i_0$ *for all* $\eta \in 2^{<\omega}$.

Proof. Let c be a map from $2^{<\omega_1}$ to ω. First we show that there exist $\eta_0 \in 2^{<\omega_1}$ and $i_0 \in \omega$ such that for all $\nu \trianglerighteq \eta_0$, there exists $\nu^* \trianglerighteq \nu$ such that $c(\nu^*) = i_0$. Suppose not. Then for all $\eta \in 2^{<\omega_1}$ and $i \in \omega$, there exists $\nu \trianglerighteq \eta$ such that $c(\nu^*) \neq i$ for all

$\nu^* \trianglerighteq \nu$. In particular, there exists $\nu_0 \trianglerighteq \langle\rangle$ such that $c(\nu^*) \neq 0$ for all $\nu^* \trianglerighteq \nu_0$. And there exists $\nu_1 \trianglerighteq \nu_0$ such that $c(\nu^*) \neq 1$ for all $\nu^* \trianglerighteq \nu_1$. For that ν_1, note that $c(\nu^*) \neq 0$ for all $\nu^* \trianglerighteq \nu_1$ since $\nu_1 \trianglerighteq \nu_0$. By repeating this process, we can find $\langle \nu_i \rangle_{i\in\omega}$ such that $\nu_{i+1} \trianglerighteq \nu_i$ for all $i \in \omega$ and $c(\nu^*) \neq i$ for all $\nu^* \trianglerighteq \nu_i$ for each $i \in \omega$. Put $\nu_\omega = \bigcup_{i\in\omega} \nu_i$. This ν_ω is an element of $2^{<\omega_1}$ by the cofinality of ω_1. But $c(\nu_\omega) \neq i$ for all $i \in \omega$ since $\nu_\omega \trianglerighteq \nu_i$. It is impossible.

Thus we have $\eta_0 \in 2^{<\omega_1}$ and $i_0 \in \omega$ such that for all $\nu \trianglerighteq \eta_0$ there exists $\nu^* \trianglerighteq \nu$ such that $c(\nu^*) = i_0$. Define $g : 2^{<\omega} \to 2^{<\omega_1}$ by

$$g(\eta) = \begin{cases} (\eta_0)^* & \text{if } \eta = \langle\rangle \\ (g(\nu)^\frown 0)^* & \text{if } \eta = \nu^\frown 0 \text{ for some } \nu \in 2^{\leq n} \\ (g(\nu)^\frown 1)^* & \text{if } \eta = \nu^\frown 1 \text{ for some } \nu \in 2^{\leq n}. \end{cases}$$

Then by the choice of η_0 and i_0, we have $c(g(\eta)) = i_0$ for all $\eta \in 2^{<\omega}$. Thus g satisfies (ii). It is also clear that g satisfies (i) since $g(\eta)^\frown l \trianglelefteq (g(\eta)^\frown l)^* = g(\eta^\frown l)$ for all $\eta \in 2^{<\omega}$ and $l \leq 1$. □

Lemma 3.7. *Let $\sigma_0, ..., \sigma_n, \sigma_{n+1}, \tau_0, ...\tau_m, \tau'_0, ..., \tau'_m \in 2^{<\omega_1}$ and assume that there exist $\eta, \nu, \theta_0, ..., \theta_m \in 2^{<\omega_1}$ such that $\tau_i = \eta^\frown\theta_i$ and $\tau'_i = \nu^\frown\theta_i$ for each $i \leq m$. If*

(i) there exists $l \leq 1$ such that $\sigma_{n+1}{}^\frown\langle l\rangle \trianglelefteq \eta, \nu$,

(ii) for that l, no $i \leq n$ satisfies $\sigma_{n+1}{}^\frown\langle l\rangle \trianglelefteq \sigma_i$,

(iii) $\sigma_{n+1}{}^\frown 1 \neq \eta$ and $\sigma_{n+1}{}^\frown 1 \neq \nu$,

then $\mathrm{cl}(\langle\sigma_0, ..., \sigma_n, \sigma_{n+1}, \tau_0, ..., \tau_m\rangle) \sim_\beta \mathrm{cl}(\langle\sigma_0, ..., \sigma_n, \sigma_{n+1}, \tau'_0, ..., \tau'_m\rangle)$.

Proof. Note that $\sigma_i \wedge \tau_j = \sigma_i \wedge \tau'_j = \sigma_i \wedge \sigma_{n+1}$ for all $i \leq n+1$, $j \leq m$, and $\sigma_{n+1}{}^\frown l \trianglelefteq \tau_i \wedge \tau_j, \tau'_i \wedge \tau'_j$ for all $i, j \leq m$. Let $\zeta_0 = \chi_0 = \sigma_0$, ... , $\zeta_{n+1} = \chi_{n+1} = \sigma_{n+1}$, and

$\zeta_{n+2+i} = \tau_i$, $\chi_{n+2+i} = \tau'_i$ for all $i \leq m$. We show $\mathrm{cl}(\overline{\zeta}) = \mathrm{cl}(\langle \zeta_0, ..., \zeta_{n+2+m} \rangle)$ and $\mathrm{cl}(\overline{\chi}) = \mathrm{cl}(\langle \chi_0, ..., \chi_{n+2+m} \rangle)$ are β-equivalent.

Clearly they satisfy (i) in Definition 2.5 since they are closed under $\wedge$. Thus we first show $\mathrm{cl}(\overline{\zeta})$ and $\mathrm{cl}(\overline{\chi})$ satisfy (ii) in Definition 2.5. Suppose $\zeta_i \wedge \zeta_j \trianglelefteq \zeta_k \wedge \zeta_l$. If $i < n+2$ or $j < n+2$, and $k < n+2$ or $l < n+2$, then $\zeta_i \wedge \zeta_j = \chi_i \wedge \chi_j$ and $\zeta_k \wedge \zeta_l = \chi_k \wedge \chi_l$ as we observed above, so that $\chi_i \wedge \chi_j \trianglelefteq \chi_k \wedge \chi_l$. If $i, j > n+1$, and $k < n+2$ or $l < n+2$, then we have $\zeta_i \wedge \zeta_j = \tau_{i-n-2} \wedge \tau_{j-n-2} \trianglerighteq \sigma_{n+1}{}^\frown l$ and $\zeta_k \wedge \zeta_l \trianglelefteq \sigma_{n+1}$ or $\zeta_k \wedge \zeta_l \trianglerighteq \sigma_{n+1}{}^\frown (l-1)^2$ as observed above, which yield a contradiction with the assumption $\zeta_i \wedge \zeta_j \trianglelefteq \zeta_k \wedge \zeta_l$. If $i < n+2$ or $j < n+2$, and $k, l > n+1$, then we have $\zeta_i \wedge \zeta_j = \chi_i \wedge \chi_j \trianglelefteq \sigma_{n+1}$ and $\chi_i \wedge \chi_j = \tau'_{i-n-2} \wedge \tau'_{j-n-2} \not\trianglerighteq \sigma_{n+1}$. Therefore $\chi_i \wedge \chi_j \trianglelefteq \chi_k \wedge \chi_l$. If $i, j > n+1$, and $k, l > n+1$, then we have $\tau_{i-n-2} \wedge \tau_{j-n-2} = \zeta_i \wedge \zeta_j \trianglelefteq \zeta_k \wedge \zeta_l = \tau_{k-n-2} \wedge \tau_{l-n-2}$, where

$$\tau_{d-n-2} = \eta{}^\frown \theta_{d-n-2}$$
$$\tau'_{d-n-2} = \nu{}^\frown \theta_{d-n-2}$$

for all $d = i, j, k, l$. Thus we have $\theta_{i-n-2} \wedge \theta_{j-n-2} \trianglelefteq \theta_{k-n-2} \wedge \theta_{l-n-2}$ and hence $\nu{}^\frown \theta_{i-n-2} \wedge \nu{}^\frown \theta_{j-n-2} \trianglelefteq \nu{}^\frown \theta_{k-n-2} \wedge \nu{}^\frown \theta_{l-n-2}$. Therefore $\chi_i \wedge \chi_j = \tau'_{i-n-2} \wedge \tau'_{j-n-2} \trianglelefteq \tau'_{k-n-2} \wedge \tau'_{l-n-2} = \chi_k \wedge \chi_l$. This shows that $\mathrm{cl}(\overline{\zeta})$ and $\mathrm{cl}(\overline{\chi})$ satisfy the second condition of β-equivalent. By the same argument, we can show $\mathrm{cl}(\overline{\zeta})$ and $\mathrm{cl}(\overline{\chi})$ satisfy (iii) in Definition 2.5.

Now we show they satisfy (iv). Suppose $(\zeta_i \wedge \zeta_j)^\frown 1 = \zeta_k \wedge \zeta_l$. By the same argument as above, we only have to check the case $i < n+2$ or $j < n+2$, and $k, l > n+1$. Then $\zeta_i \wedge \zeta_j \trianglelefteq \sigma_{n+1}$ or $\zeta_i \wedge \zeta_j \trianglerighteq \sigma_{n+1}{}^\frown (l-1)^2$. But it is not possible that $\zeta_i \wedge \zeta_j \trianglerighteq \sigma_{n+1}{}^\frown (l-1)^2$ since $\zeta_k \wedge \zeta_l \trianglerighteq \sigma_{n+1}{}^\frown l$. Thus we have $\zeta_i \wedge \zeta_j \trianglelefteq \sigma_{n+1} \trianglelefteq \sigma_{n+1}{}^\frown l \trianglelefteq \zeta_k \wedge \zeta_l = (\zeta_i \wedge \zeta_j)^\frown 1$. Therefore $l = 1$ and $\sigma_{n+1}{}^\frown 1 = \zeta_k \wedge \zeta_l$. But then we

have

$$\sigma_{n+1}{}^\frown 1 \trianglelefteq \eta \trianglelefteq \eta^\frown(\theta_{i-n-2} \wedge \theta_{j-n-2}) = \zeta_k \wedge \zeta_l = \sigma_{n+1}{}^\frown 1,$$

and hence $\sigma_{n+1}{}^\frown 1 = \eta$. This violate the choice of η. Thus if $i < n+2$ or $j < n+2$, and $k, l > n+1$, then $(\zeta_i \wedge \zeta_j)^\frown 1 \neq \zeta_k \wedge \zeta_l$ and similarly $(\chi_i \wedge \chi_j)^\frown 1 \neq \chi_k \wedge \chi_l$. This shows that cl$(\overline{\zeta})$ and cl$(\overline{\chi})$ satisfy (iv) in Definition 2.5. □

Now we are ready to construct an antichain tree. Note that if T has NSOP$_2$ and SOP$_1$, then the hypothesis of Theorem 3.8 below always holds.

Theorem 3.8. *Suppose there exists $\varphi(x, y)$ which witnesses SOP$_1$ and there is no $n \in \omega$ such that $\bigwedge_{i=0}^n \varphi(x, y_i)$ witnesses SOP$_2$. Then there exists $\langle b_\eta \rangle_{\eta \in 2^{<\omega}}$ such that $\langle \varphi(x, y), \langle b_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ forms an antichain tree.*

Proof. By Theorem 2.11 and compactness, there exists a β-indiscernible $\langle a_\eta \rangle_{\eta \in 2^{<\omega_1}}$ which witnesses SOP$_1$ with φ. Define a map $h : 2^{<\omega} \to 2^{<\omega}$ by

$$h(\eta) = \begin{cases} 1 & \text{if } \ \eta = \langle\rangle \\ h(\eta^-)^\frown\langle 001\rangle & \text{if } \ t(\eta) = 0 \\ h(\eta^-)^\frown\langle 011\rangle & \text{if } \ t(\eta) = 1. \end{cases}$$

For each $i, k \in \omega$ and $\eta, \xi \in 2^{<\omega_1}$, put

$$\begin{aligned} L_i &= \{h(\nu') : l(\nu') = i\}, & L_i(\eta) &= \{\eta^\frown\nu : \nu \in L_i\}, \\ 1_\xi &= \{\xi^\frown 1^d : d \in \omega\}, & 1_\xi(\eta) &= \{\eta^\frown\nu : \nu \in 1_\xi\}, \\ 1^k_\xi &= \{\xi^\frown 1^0, ..., \xi^\frown 1^k\}, & 1^k_\xi(\eta) &= \{\eta^\frown\nu : \nu \in 1^k_\xi\}, \\ M_i &= L_i \cup 1_{h(0^i)}, & M_i(\eta) &= \{\eta^\frown\nu : \nu \in M_i\}, \\ M^k_i &= L_i \cup 1^k_{h(0^i)}, & M^k_i(\eta) &= \{\eta^\frown\nu : \nu \in M^k_i\}, \\ m^k_i &= h(0^i)^\frown 1^k, & m^k_i(\eta) &= \eta^\frown(m^k_i), \end{aligned}$$

For each $X \subseteq 2^{<\omega_1}$, we define Φ_X by $\{\varphi(x, a_\eta) : \eta \in X\}$.

Let us note some remarks in advance to make the proof easier.

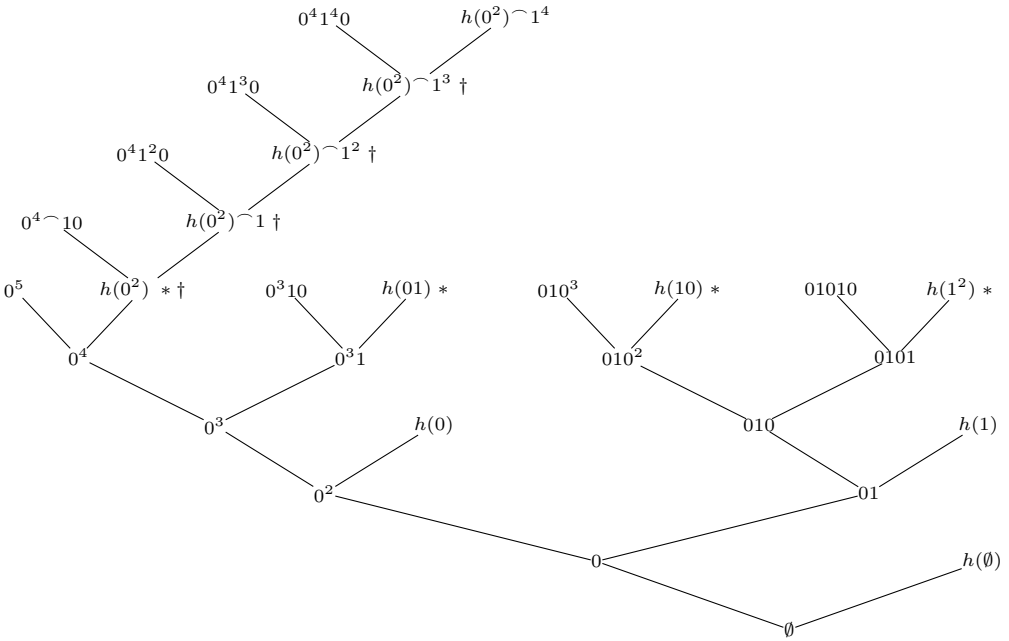


Figure 3.8.1: $m_2^3 = h(0^2)^\frown 1^3$ and $M_2^3 = L_2 \cup 1^3_{h(0^2)}$, where L_2 is the set of elements marked *, $1^3_{h(0^2)}$ is the set of elements marked †.

- $L_0 = \{1\}$.
- $M_0 = 1_{\langle 1 \rangle} \subseteq 1_{\langle\rangle}$.
- $M_0(\eta) = 1_1(\eta) \subseteq 1_{\langle\rangle}(\eta) = 1_\eta$.
- $m_i^k \in M_i^k$ for all $i, k \in \omega$.
- m_i^k is the longest element of M_i^k for all $i, j \in \omega$.
- $t(\eta) = 1$ for all $\eta \in M_i^k$ for all $i, k \in \omega$.
- $\bigwedge M_i^k = 0$ for all $i, k \in \omega$ with $i > 0$.
- $\bigwedge M_0^k = 1$ for all $k \in \omega$.

Claim. There exists $\eta \in 2^{<\omega_1}$ such that $\Phi_{M_i(\eta)}$ is consistent for all $i \in \omega$.

Proof. Suppose not. We construct $\langle w_\eta, u_\eta \rangle_{\eta \in 2^{<\omega_1}}$ which satisfies

(i) $w_\eta \subseteq 2^{<\omega_1}$, $u_\eta \in 2^{<\omega_1}$ for all $\eta \in 2^{<\omega_1}$,

(ii) $w_{\langle\rangle} = \emptyset$, $u_{\langle\rangle} = \langle 110 \rangle$,

(iii) for all $\eta \in 2^{<\omega_1}$ and $l \leq 1$, there exist $i, k \in \omega$ such that

$$w_{\eta^\frown l} = M_i^k((u_\eta)^\frown l),\ u_{\eta^\frown l} = m_i^k((u_\eta)^\frown l)^\frown 110,$$

(iv) if $l(\eta)$ is a limit, then $u_\eta = (\bigcup_{\nu \precneqq \eta} u_\nu)^\frown 110$, $w_\eta = \emptyset$, and $l(u_\eta)$ is limit,

(v) $\Phi_{p_\eta \cup w_{\eta^\frown 0} \cup w_{\eta^\frown 1}}$ is inconsistent for all $\eta \in 2^{<\omega_1}$, where $p_\eta = \bigcup_{\nu \trianglelefteq \eta} w_\nu$,

(vi) for all $\eta, \nu \in 2^{<\omega_1}$, if $\eta \trianglelefteq \nu$ then $u_\eta \precneqq (u_\nu)^{---}$,

(vii) there is no $\xi \in p_\eta$ such that $(u_\eta)^{---} \precneqq \xi$,

(viii) for all $\eta, \nu \in 2^{<\omega_1}$ and $l \leq 1$, if $\eta^\frown l \trianglelefteq \nu$ then $u_\eta{}^\frown l \trianglelefteq \xi$ for all $\xi \in w_\nu$,

(ix) $\Phi_{p_\eta \cup 1_{(u_\eta)^-}}$ is consistent for all $\eta \in 2^{<\omega_1}$.

First we put $w_{\langle\rangle} = \emptyset$ and $u_{\langle\rangle} = \langle 110 \rangle$. Then clearly $\Phi_{p_{\langle\rangle} \cup 1_{(u_{\langle\rangle})^-}}$ is consistent. Suppose that we have constructed w_ν and u_ν for all $\nu \trianglelefteq \eta$. By (ix), $\Phi_{p_\eta \cup 1_{(u_\eta)^-}}$ is consistent so that its subset $\Phi_{p_\eta \cup M_0((u_\eta)^-)}$ is also consistent. Since we assume that $\Phi_{M_i((u_\eta)^-)}$ is not consistent for some $i \in \omega$, there exists $i \in \omega$ such that $\Phi_{p_\eta \cup M_i((u_\eta)^-)}$ is consistent and $\Phi_{p_\eta \cup M_{i+1}((u_\eta)^-)}$ is inconsistent. Note that $t(u_\nu) = 0$ for all $\nu \trianglelefteq \eta$ and $M_{i+1} \subseteq M_i(00) \cup M_i(01)$. Thus we have

$$\begin{aligned} M_{i+1}((u_\eta)^-) \subseteq\ & M_i((u_\eta)^-{}^\frown 00) \cup M_i((u_\eta)^-{}^\frown 01) \\ =\ & M_i((u_\eta)^\frown 0) \cup M_i((u_\eta)^\frown 1). \end{aligned}$$

Thus

$$\Phi_{p_\eta \cup M_i((u_\eta)^\frown 0) \cup M_i((u_\eta)^\frown 1)}$$

is inconsistent. By compactness, there exists $k \in \omega$ such that

$$\Phi_{p_\eta \cup M_i^k((u_\eta)^\frown 0) \cup M_i^k((u_\eta)^\frown 1)}$$

is inconsistent. Put

$$\begin{aligned} w_{\eta^\frown 0} &= M_i^k((u_\eta)^\frown 0), & w_{\eta^\frown 1} &= M_i^k((u_\eta)^\frown 1), \\ u_{\eta^\frown 0} &= (m_i^k((u_\eta)^\frown 0))^\frown 110, & u_{\eta^\frown 1} &= (m_i^k((u_\eta)^\frown 1))^\frown 110. \end{aligned}$$

Then $w_{\eta^\frown 0}$, $w_{\eta^\frown 1}$, $u_{\eta^\frown 0}$, $u_{\eta^\frown 1}$ satisfy (i)-(viii). To show they satisfy (ix), note that

$$\begin{aligned} p_{\eta^\frown l} \cup 1_{(u_{\eta^\frown l})^-} &= p_\eta \cup w_{\eta^\frown l} \cup 1_{(u_{\eta^\frown l})^-} \\ &= p_\eta \cup M_i^k((u_\eta)^\frown l) \cup 1_{(m_i^k((u_\eta)^\frown l))^\frown 11} \\ &\subseteq p_\eta \cup M_i((u_\eta)^\frown l). \end{aligned}$$

Thus it is enough to show that $\Phi_{p_\eta \cup M_i((u_\eta)^\frown l)}$ is consistent. Choose any finite subset of $p_\eta \cup M_i((u_\eta)^\frown l)$. This finite subset can be regarded as a tuple of the form

$$\langle \sigma_0, ..., \sigma_n, (u_\eta)^\frown l^\frown \tau_0, ..., (u_\eta)^\frown l^\frown \tau_m \rangle,$$

where $\sigma_0, ..., \sigma_n \in p_\eta$ and $\tau_0, ..., \tau_m \in M_i$. Say $\bar{\sigma} = \langle \sigma_0, ..., \sigma_n \rangle$ and $\bar{\tau} = \langle (u_\eta)^\frown l^\frown \tau_0$, ..., $(u_\eta)^\frown l^\frown \tau_m \rangle$. Define a tuple $\bar{\tau}'$ by

$$\bar{\tau}' = \langle (u_\eta)^-{}^\frown \tau_0, ..., (u_\eta)^-{}^\frown \tau_m \rangle.$$

Then $\bar{\tau}' \subseteq M_i((u_\eta)^-)$. Recall that $\Phi_{p_\eta \cup M_i((u_\eta)^-)}$ is consistent. Thus $\Phi_{\bar{\sigma}^\frown \bar{\tau}'}$ is consistent. Thus it is enough to show that $\mathrm{cl}(\bar{\sigma}^\frown \bar{\tau}) \sim_\beta \mathrm{cl}(\bar{\sigma}^\frown \bar{\tau}')$. By the construction and induction hypothesis, we have

- $(u_\eta)^{---}{}^\frown 11 \trianglelefteq u_\eta^-, (u_\eta)^\frown l$,

- there is no $i \leq n$ such that $(u_\eta)^{---} \trianglelefteq \sigma_i$.

Thus $\mathrm{cl}(\bar{\sigma}^\frown(u_\eta)^{---}{}^\frown\bar{\tau}) \sim_\beta \mathrm{cl}(\bar{\sigma}^\frown(u_\eta)^{---}{}^\frown\bar{\tau}')$ by Lemma 3.7. In particular we have $\mathrm{cl}(\bar{\sigma}^\frown\bar{\tau}) \sim_\beta \mathrm{cl}(\bar{\sigma}^\frown\bar{\tau}')$ as desired. Hence $w_{\eta^\frown l}, u_{\eta^\frown l}$ satisfies (i)-(ix).

For the limit case, suppose $l(\eta)$ is a limit ordinal, w_ν and u_ν are constructed for all $\nu \trianglelefteq \eta$. Put $u_\eta = (\bigcup_{\nu \trianglelefteq \eta} u_\nu)^\frown\langle 110\rangle$ and $w_\eta = \emptyset$. Then by compactness and β-indiscernibility, it is clear that w_η and u_η satisfy (ix). (i)-(viii) are clear. This completes the choice of $\langle w_\eta, u_\eta\rangle_{\eta\in 2^{<\omega_1}}$. Note that

- for all $\eta, \nu \in 2^{<\omega_1}$ and $l \leq 1$, if $\eta^\frown l \trianglelefteq \nu$ then $u_\eta{}^\frown l \trianglelefteq u_\nu$.

For each $\eta \in 2^{<\omega_1}$, there exists a finite subset of p_η which is inconsistent with $w_{\eta^\frown 0} \cup w_{\eta^\frown 1}$. Let us call this finite subset q_η.

Subclaim 1. There exists a finite subset q of $2^{<\omega_1}$ and $\eta^* \in 2^{<\omega_1}$ such that for any $\eta \trianglerighteq \eta^*$, there exists $\eta' \trianglerighteq \eta$ such that $q_{\eta'} = q$.

Proof. Suppose not. Then

($*$) for any finite $q \subseteq 2^{<\omega_1}$ and $\eta \in 2^{<\omega_1}$ there exists $\eta^q \trianglerighteq \eta$ such that $q_{\eta'} \neq q$ for all $\eta' \trianglerighteq \eta^q$.

Subsubclaim. For all $\eta \in 2^{<\omega_1}$, there exists $\eta' \trianglerighteq \eta$ such that $q_{\eta''} \not\subseteq p_\eta$ for all $\eta'' \trianglerighteq \eta'$.

Proof. Suppose not. Then

($\dagger$) there exists $\eta \in 2^{<\omega_1}$ such that for all $\eta' \trianglerighteq \eta$ there is some $\eta'' \trianglerighteq \eta'$ such that $q_{\eta''} \subseteq p_\eta$.

For that η, there exists $\eta^{\emptyset} \trianglerighteq \eta$ such that $q_{\eta'} \neq \emptyset$ for all $\eta' \trianglerighteq \eta^{\emptyset}$ by $(*)$. And by the choice of η in $(\dagger)$, there exists $\eta_1 \trianglerighteq \eta^{\emptyset}$ such that $q_{\eta_1} \subseteq p_\eta$. Similarly we can choose $\eta^{q_{\eta_1}} \trianglerighteq \eta_1$ such that $q_{\eta'} \neq q_{\eta_1}$ for all $\eta' \trianglerighteq \eta^{q_{\eta_1}}$ by $(*)$. And there exists $\eta_2 \trianglerighteq \eta^{q_{\eta_1}}$ such that $q_{\eta_2} \subseteq p_\eta$. By iterating this process, one can choose $\langle \eta_i \rangle_{i \in \omega_1}$ which satisfies $q_{\eta_i} \subseteq p_\eta$ for all $i \in \omega_1$, and $q_i \neq q_j$ for all $i, j \in \omega_1$. But it is not possible since the number of finite subsets of countable set p_η is countable. This proves Subsubclaim. $\square$

Hence for all $\eta \in 2^{<\omega_1}$ there exists $\eta' \trianglerighteq \eta$ such that $q_{\eta''} \not\subseteq p_\eta$ for all $\eta'' \trianglerighteq \eta'$. Put $\eta_0 = \langle\rangle$. And recursively choose $\eta_{i+1} \trianglerighteq \eta_i$ which satisfies $q_{\eta''} \not\subseteq p_{\eta_i}$ for all $\eta'' \trianglerighteq \eta_{i+1}$. Let η_ω be $\bigcup_{i \in \omega} \eta_i$. Then $p_{\eta_\omega} = \bigcup_{i \in \omega} p_{\eta_i}$. Since q_{η_ω} is finite, there exists $i \in \omega$ such that $q_{\eta_\omega} \subseteq p_{\eta_i}$. Since $\eta_\omega \trianglerighteq \eta_{i+1}$, we have $q_{\eta_\omega} \not\subseteq p_{\eta_i}$, a contradiction. This proves Subclaim 1. $\square$

Now we are ready to prove the claim. To show the claim, we construct a witness of SOP_2. Fix a finite subset q of $2^{<\omega_1}$ and $\eta^* \in 2^{<\omega_1}$ satisfying Subclaim 1. By the choice of q, η^*, and cofinality of ω_1 we can find a map $f : 2^{<\omega_1} \to 2^{<\omega_1}$ such that

- $f(\emptyset) = \eta^*$,
- $f(\eta) \trianglelefteq f(\nu)$ for all $\eta, \nu \in 2^{<\omega_1}$ with $\eta \trianglelefteq \nu$,
- $f(\eta)^\frown l \trianglelefteq f(\eta^\frown l)$ for all $\eta \in 2^{<\omega_1}$, $l \leq 1$,
- $f(\eta) \trianglerighteq \bigcup_{\nu \trianglelefteq \eta} f(\nu)$ for all η such that $l(\eta)$ is a limit ordinal,
- $q_{f(\eta)} = q$ for all $\eta \in 2^{<\omega_1}$.

Note that for each $\eta \in 2^{<\omega_1}$, there exists $i, k \in \omega$ such that $w_{f(\eta)^\frown l}$ is of the form $M_i^k((u_{f(\eta)})^\frown l)$ for all $l \leq 1$. By using Lemma 3.6, we can find a map $g : 2^{<\omega} \to 2^{<\omega_1}$

and $i, k \in \omega$ such that

- $g(\eta)^\frown\langle l\rangle \trianglelefteq g(\eta^\frown\langle l\rangle)$ for all $\eta \in 2^{<\omega}$ and $l \leq 1$,
- $w_{f(g(\eta))^\frown\langle l\rangle} = M_i^k((u_{f(g(\eta))})^\frown\langle l\rangle)$ for for all $\eta \in 2^{<\omega}$ and $l \leq 1$.

Let $j = |q|$ and $e = j + |M_i^k|$, so that $e = j + k + 2^i$. Say $q = \{\mu_0, ..., \mu_{j-1}\}$ and $M_i^k = \{\mu_j, ..., \mu_{e-1}\}$. Then $M_i^k(\lambda) = \{\lambda^\frown\mu_j, ..., \lambda^\frown\mu_{e-1}\}$ for all $\lambda \in 2^{<\omega_1}$. Put $\psi(x, y_0, ..., y_{e-1}) = \varphi(x, y_0) \wedge ... \wedge \varphi(x, y_{e-1})$ and denote it by $\psi(x, \bar{y})$. For each $\eta \in 2^{<\omega}$ with $l(\eta) > 0$, let $\lambda_\eta = (u_{f(g(\eta^-))})^\frown\langle t(\eta)\rangle$ and

$$\bar{a}_\eta = \langle a_{\mu_0}, ..., a_{\mu_{j-1}}, a_{(\lambda_\eta)^\frown\mu_j}, ..., a_{(\lambda_\eta)^\frown\mu_{e-1}}\rangle,$$
$$z(\eta) = \langle \mu_0, ..., \mu_{j-1}, (\lambda_\eta)^\frown\mu_j, ..., (\lambda_\eta)^\frown\mu_{e-1}\rangle.$$

Note that $z(\eta) = q \cup w_{f(g(\eta^-))^\frown\langle t(\eta)\rangle}$.

Subclaim 2. $\{\psi(x, \bar{a}_\eta), \psi(x, \bar{a}_\nu)\}$ is inconsistent for any $\eta, \nu \in 2^{<\omega}$ with $\eta \perp \nu$ and $0 \trianglelefteq \eta, \nu$. $\{\psi(x, \bar{a}_{\eta\restriction n}) : 0 < n \in \omega\}$ is consistent for all $\eta \in 2^{<\omega}$ with $0 \trianglelefteq \eta$.

Proof. Suppose $\eta \perp \nu$ and $0 \trianglelefteq \eta, \nu$. Say $\xi = \eta \wedge \nu$. Clearly $0 \trianglelefteq \xi$ and we may assume $\xi^\frown 0 \trianglelefteq \eta$ and $\xi^\frown 1 \trianglelefteq \nu$. Then $f(g(\xi))^\frown 0 \trianglelefteq f(g(\eta^-))^\frown\langle t(\eta)\rangle$ and $f(g(\xi))^\frown 1 \trianglelefteq f(g(\eta^-))^\frown\langle t(\mu)\rangle$ so that $u_{f(g(\xi))}{}^\frown 0 \trianglelefteq u_{f(g(\xi))^\frown 0} \trianglelefteq \lambda_\eta$ and $u_{f(g(\xi))}{}^\frown 1 \trianglelefteq u_{f(g(\xi))^\frown 1} \trianglelefteq \lambda_\nu$. Since $\nu_s \in q = q_{f(g(\xi))} \subseteq p_{f(g(\xi))}$ for each $s < j$, we have $(u_{f(g(\xi))}) \not\trianglelefteq \mu_s$ in particular $(u_{f(g(\xi))})^\frown 0 \not\trianglelefteq \mu_s$ for all $s < j$. It is clear that $u_{f(g(\xi))}{}^\frown 0 \trianglelefteq \lambda_{\xi^\frown 0}$. Thus we can apply Lemma 3.7 to obtain

$$\mathrm{cl}(z(\eta)z(\nu)) \sim_\beta \mathrm{cl}(z(\xi^\frown 0)z(\nu)).$$

As we remarked, $\bigwedge M_i^k = \langle l\rangle$ for some $l \leq 1$. Thus $\mu_s = \langle l\rangle^\frown\mu_s'$ for some $\mu_s' \in 2^{<\omega_1}$ for each $j \leq s < e$. Thus

$$\begin{aligned}(\lambda_{\xi^\frown 1})^\frown\mu_s = (\lambda_{\xi^\frown 1})^\frown\langle l\rangle^\frown\mu_s' = &\quad (u_{f(g(\xi))})^\frown 1^\frown\langle l\rangle^\frown\mu_s',\\ (\lambda_\nu)^\frown\mu_s = &\quad (\lambda_\nu)^\frown\langle l\rangle^\frown\mu_s'\end{aligned}$$

for all $j \leq s < e$. Clearly

$$\begin{aligned}
&(u_{f(g(\xi))})^\frown 1 \not\trianglelefteq \lambda_{\xi^\frown 0},\\
&(u_{f(g(\xi))})^\frown 1 \not\trianglelefteq \mu_s,\\
&(u_{f(g(\xi))})^\frown 1 \neq (u_{f(g(\xi))})^\frown 1^\frown \langle l\rangle,\\
&(u_{f(g(\xi))})^\frown 1 \neq (\lambda_\nu)^\frown \langle l\rangle
\end{aligned}$$

for all $s < j$. By Lemma 3.7 again, we have

$$\mathrm{cl}(z(\xi^\frown 0)z(\nu)) \sim_\beta \mathrm{cl}(z(\xi^\frown 0)z(\xi^\frown 1)).$$

As we remarked above

$$\begin{aligned}
z(\xi^\frown 0) &= q \cup w_{f(g(\xi))^\frown 0},\\
z(\xi^\frown 1) &= q \cup w_{f(g(\xi))^\frown 1}.
\end{aligned}$$

Hence $\{\psi(x, \bar{a}_{\xi^\frown 0}), \psi(x, \bar{a}_{\xi^\frown 1})\}$ is inconsistent as $\Phi_{q \cup w_{f(g(\xi))^\frown 0} \cup w_{f(g(\xi))^\frown 1}}$ is inconsistent. By β-indiscernibility $\{\psi(x, \bar{a}_\eta), \psi(x, \bar{a}_\nu)\}$ is also inconsistent.

Suppose $\eta \in 2^\omega$. Consistency of $\{\psi(x, \bar{a}_{\eta\restriction n}) : 0 < n \in \omega\}$ follows the consistency of Φ_{p_η}. This proves Subclaim 2. □

Hence ψ witnesses SOP_2. But we assume that any conjunction of φ does not witness SOP_2. Thus we have a contradiction. This proves Claim. □

Therefore, there exists $\eta \in 2^{<\omega_1}$ such that $\Phi_{M_i(\eta)}$ is consistent for all $i \in \omega$. By β-indiscernibility, we may assume $\eta = \langle\rangle$. For each $\eta \in 2^{<\omega}$, put $b_\eta = a_{h(\eta)}$. We show that $\langle \varphi(x, b_\eta)\rangle_{\eta \in 2^{<\omega}}$ is an antichain tree. Suppose $\eta \trianglelefteq \nu$. It easily follows that $h(\eta)^{-\frown} 1 = h(\eta)$ and $h(\eta)^{-\frown} 0 \trianglelefteq h(\nu)$ and hence $\{\varphi(x, b_\eta), \varphi(x, b_\nu)\}$ is inconsistent. Thus if $X \subseteq 2^{<\omega}$ is not an antichain, then the set $\{\varphi(x, b_\eta) : \eta \in X\}$ is not consistent.

Let $\{\eta_0, ..., \eta_n\} \subseteq 2^{<\omega}$ be an antichain. Then there exist $\nu_0, ..., \nu_n$ such that $\eta_i \trianglelefteq \nu_i$ for all $i \leq n$, and $l(\nu_0) = ... = l(\nu_n)$. Hence $\{\varphi(x, b_{\nu_0}), ..., \varphi(x, b_{\nu_n})\}$ is consistent

since $h(\nu_0), ..., h(\nu_n) \in M_{l(\nu_0)}$. Thus the only remain is to show that $\mathrm{cl}(\langle h(\eta_0), ..., h(\eta_n)\rangle)$ $\sim_\beta \mathrm{cl}(\langle h(\nu_0), ..., h(\nu_n)\rangle)$.

First, we may assume that $\eta_0, ..., \eta_n$ are ordered by lexicographic order. That is, $(\eta_i \wedge \eta_{i+1})^\frown 0 \trianglelefteq \eta_i$ for all $i < n$. It is easy to show that for any $i < j \leq n$,

- $h(\eta_i) \wedge h(\eta_j) = h(\nu_i) \wedge h(\eta_j) = h(\eta_i) \wedge h(\nu_j) = h(\nu_i) \wedge h(\nu_j)$,
- $h(\eta_i) \wedge h(\eta_j)^\frown 0 \trianglelefteq h(\eta_i), h(\nu_i)$,
- $(h(\eta_i) \wedge h(\eta_j))^\frown 1 \trianglelefteq h(\eta_j), h(\nu_j)$,
- $(h(\eta_i) \wedge h(\eta_j))^\frown 1 \neq h(\eta_j), h(\nu_j)$.

Therefore, we can apply Lemma 3.7 repeatedly to obtain

$$\begin{aligned} & \mathrm{cl}(\langle h(\eta_0), h(\eta_1), h(\eta_2), ..., h(\eta_n)\rangle) \\ \sim_\beta\ & \mathrm{cl}(\langle h(\nu_0), h(\eta_1), h(\eta_2), ..., h(\eta_n)\rangle) \\ \sim_\beta\ & \mathrm{cl}(\langle h(\nu_0), h(\nu_1), h(\eta_2), ..., h(\eta_n)\rangle) \\ & \cdots \\ \sim_\beta\ & \mathrm{cl}(\langle h(\nu_0), h(\nu_1), h(\nu_2), ..., h(\nu_n)\rangle). \end{aligned}$$

This completes proof of Theorem 3.8. □

Corollary 3.9. *If* T *is* SOP_1 *and* NSOP_2*, then* T *has ATP. The witness of ATP can be selected to be strongly indiscernible.*

Proof. If a theory has SOP_1 and does not have SOP_2, then the theory has a formula which witnesses SOP_1 and any finite conjunction of the formula does not witness SOP_2. Thus we can apply Theorem 3.8. The theory has a witness of ATP. Furthermore, we can obtain a strongly indiscernible witness of ATP by using compactness and the modeling property. □

3.3 Opposing natures of ATP and SOP_2

As we observed in the beginning of this chapter, one can find witnesses of SOP_1 and TP_2 from an antichain tree $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ by restricting the parameter part $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$. But we can not use the same method for finding a witness of SOP_2.

Remark 3.10. ATP and SOP_2 do not contain each other in the following sense.

(i) Suppose $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ is an antichain tree. Then there is no $h : 2^{\leq 2} \to 2^{<\omega}$ such that $\langle \varphi(x,y), \langle b_\eta \rangle_{\eta \in 2^{\leq 2}} \rangle$ satisfies the conditions of SOP_2, where $b_\eta = a_{h(\eta)}$ for each $\eta \in 2^{\leq 2}$.

(ii) Suppose $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ witnesses SOP_2. Then there is no $h : 2^{\leq 2} \to 2^{<\omega}$ such that $\langle \varphi(x,y), \langle b_\eta \rangle_{\eta \in 2^{\leq 2}} \rangle$ forms an antichain tree with height 2, where $b_\eta = a_{h(\eta)}$ for each $\eta \in 2^{\leq 2}$.

Proof. (i) To get a contradiction, suppose there exists such h. Then $h(00), h(01), h(10)$, and $h(11)$ are pairwisely comparable in $2^{<\omega}$, hence they are linearly ordered by $\trianglelefteq$. We may assume $h(00)$ is the smallest. Since $h(0)$ and $h(00)$ are incomparable, $h(0)$ and $h(11)$ are incomparable. Thus $\{\varphi(x,b_0), \varphi(x,b_{11})\}$ is consistent. This is a contradiction.

(ii) To get a contradiction, suppose there exists such h. Then $h(00)$, $h(01)$, $h(10)$, and $h(11)$ are pairwisely comparable in $2^{<\omega}$, hence they are linearly ordered by $\trianglelefteq$. We may assume $h(00)$ is the smallest. Since $h(0)$ and $h(00)$ are incomparable, $h(0)$ and $h(11)$ are incomparable. Thus $\{\varphi(x,b_0), \varphi(x,b_{11})\}$ is inconsistent. This is a contradiction. □

But it does not mean the existence of an antichain tree prevents the theory from having a witness of SOP_2. Under some additional conditions, the existence of an antichain tree is still likely to cause SOP_2. For example, if $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ forms an antichain tree and '$\trianglelefteq$' is definable by some formula, then $\trianglelefteq$ could be a witness of SOP_2. We will consider this little more in Chapter 5.

We end this chapter with the following remarks.

Remark 3.11. If the existence of an antichain tree always implies the existence of a witness of SOP_2, then SOP_1 and SOP_2 are the same by Corollary 3.9.

Remark 3.12. If there exists a NSOP_2 theory having an antichain tree, then the class of all SOP_2 theories is a proper subclass of the class of all SOP_1 theories by Proposition 3.4.

Chapter 4

1-strictly strong order property

4.1 Maximal and minimal consistency of SOP_1 witnesses

As we mentioned in the introduction, in [4] M. Džamonja and S. Shelah claimed that if there is a theory which is SOP_1-NSOP_2, then the theory has a witness of SOP_1 which satisfies certain condition (see [4, Claim 2.19], or Claim 1.10 in this thesis). Although the claim was not correct, we can obtain some results from it by modifying its condition. In this Chapter, we introduce a notion of *1-strictly strong order property ($SSOP_1$)* and correct Claim 1.10 to make sense by using SSOP_1.

Definition 4.1. We say a tuple $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ witnesses *1-strictly strong order property ($SSOP_1$)* if for all $X \subseteq 2^{<\omega}$, $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent if and only if X does not contain $\eta^\frown 1$ and $\eta^\frown 0^\frown \nu$ for some $\eta, \nu \in 2^{<\omega}$. We say a theory has SSOP_1 if it has an SSOP_1 formula. Otherwise, we say T is NSSOP_1.

Clearly if a theory has SSOP_1, then it has SOP_1. Let us compare SOP_1, SOP_2 and SSOP_1 briefly.

Remark 4.2. The notions of tree properties can be understood to have the following characteristics.

(i) Let $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ be a witness of SOP_1. Then for any $X \subseteq 2^{<\omega}$,

a. if X is linearly ordered by $\trianglelefteq$, then $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent,

b. if $\eta^\frown 1, \eta^\frown 0^\frown \nu \in X$ for some $\eta, \nu \in 2^{<\omega}$, then $\{\varphi(x, a_\eta) : \eta \in X\}$ is inconsistent,

c. if X is not linearly ordered by $\trianglelefteq$ and there do not exist $\eta, \nu \in 2^{<\omega}$ such that $\eta^\frown 1, \eta^\frown 0^\frown \nu \in X$, then the definition of SOP_1 does not decide whether the set $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent or inconsistent.

(ii) Let $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ be a witness of SOP_2. Then for any $X \subseteq 2^{<\omega}$,

a. if X is linearly ordered by $\trianglelefteq$, then $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent,

b. if $\eta^\frown 1, \eta^\frown 0^\frown \nu \in X$ for some $\eta, \nu \in 2^{<\omega}$, then $\{\varphi(x, a_\eta) : \eta \in X\}$ is inconsistent,

c. if X is not linearly ordered by $\trianglelefteq$ and there do not exist $\eta, \nu \in 2^{<\omega}$ such that $\eta^\frown 1, \eta^\frown 0^\frown \nu \in X$, then $\{\varphi(x, a_\eta) : \eta \in X\}$ is inconsistent.

(iii) Let $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ be a witness of SSOP_1. Then for any $X \subseteq 2^{<\omega}$,

a. if X is linearly ordered by $\trianglelefteq$, then $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent,

b. if $\eta^\frown 1, \eta^\frown 0^\frown \nu \in X$ for some $\eta, \nu \in 2^{<\omega}$, then $\{\varphi(x, a_\eta) : \eta \in X\}$ is inconsistent,

c. if X is not linearly ordered by $\trianglelefteq$ and there do not exist $\eta, \nu \in 2^{<\omega}$ such that $\eta^\frown 1, \eta^\frown 0^\frown \nu \in X$, then $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent.

Thus, roughly speaking, notions of SOP_2 and SSOP_1 can be considered to be obtained by removing the ambiguity of SOP_1, in the opposite way to each other (by 'am-

biguity' we mean the situation that there is a subset X of $2^{<\omega}$ such that we cannot know whether $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent or inconsistent by the definition of SOP_1 alone).

4.2 Preserving lemma for SSOP_1

As it does to SOP_1 in Theorem 2.11, β-indiscernibility preserves SSOP_1 as follows.

Proposition 4.3. *Suppose $\varphi(x, y)$ witnesses SSOP_1. Then there is a β-indiscernible tree which witnesses SSOP_1 with φ.*

Proof. We use the same argument which appears in Lemma 2.9 and Theorem 2.11. Suppose there exists $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ which witnesses SSOP_1 with φ and let $b_\eta = a_{\eta^\frown 0} a_{\eta^\frown 1}$ for each $\eta \in 2^{<\omega}$. By the modeling property of α-indiscernibility, there exists an α-indiscernible tree $\langle c_\eta \rangle_{\eta \in 2^{<\omega}}$ such that for any $\bar{\eta}$ and finite subset Δ of $\mathcal{L}$-formulas, $\bar{\nu} \sim_\alpha \bar{\eta}$ and $\bar{b}_{\bar{\nu}} \equiv_\Delta \bar{c}_{\bar{\eta}}$ for some $\bar{\nu}$. Then c_η can be written of the form $c^0_\eta c^1_\eta$ for each $\eta \in 2^{<\omega}$. For each $\eta \in 2^{<\omega}$ with $l(\eta) \geq 1$, we define d'_η by $c^0_{\eta^-}$ if $t(\eta) = 0$, $c^1_{\eta^-}$ if $t(\eta) = 1$. Put $d_\eta = d'_{0^\frown \eta}$ for each $\eta \in 2^{<\omega}$. Define ψ_0 and ψ_1 by $\psi_0(x; \bar{y_0}, \bar{y_1}) = \varphi(x, \bar{y_0}) \wedge \bar{y_1} = \bar{y_1}$ and $\psi_1(x; \bar{y_0}, \bar{y_1}) = \varphi(x, \bar{y_1}) \wedge \bar{y_0} = \bar{y_0}$ respectively.

As we observed in Lemma 2.9, $\langle d_\eta \rangle_{\eta \in 2^{<\omega}}$ is γ-indiscernible and $\{\varphi(x, d_{\eta^\frown 1}), \varphi(x, d_{\eta^\frown 0^\frown \nu})\}$ is inconsistent for all $\eta, \nu \in 2^{<\omega}$.

Claim. For all $X \subseteq 2^{<\omega}$, if there are no $\eta, \nu \in 2^{<\omega}$ such that $\eta^\frown 1, \eta^\frown 0^\frown \nu \in X$, then $\{\varphi(x, d_\xi) : \xi \in X\}$ is consistent.

Proof. By compactness, we may assume X is finite. Let $X_0 = \{\xi \in X : t(\xi) = 0\}$, $X_1 = \{\xi \in X : t(\xi) = 1\}$. And let $\{\sigma_0{}^\frown 0, ..., \sigma_n{}^\frown 0\}$, $\{\tau_0{}^\frown 1, ..., \tau_m{}^\frown 1\}$ denote X_0,

X_1 respectively. Then $\{\varphi(x, d_\xi) : \xi \in X\}$ is equal to

$$\{\psi_0(x, c_{\sigma_0}), ..., \psi_0(x, c_{\sigma_n})\} \cup \{\psi_1(x, c_{\tau_0}), ..., \psi_1(x, c_{\tau_n})\}.$$

Since there are no $\eta, \nu \in 2^{<\omega}$ such that $\eta^\frown 1, \eta^\frown 0^\frown \nu \in X$, we have

(i) $\sigma_i \neq \tau_j$ for all $i \leq n$ and $j \leq m$,

(ii) $\sigma_i \not\trianglerighteq \tau_j^\frown 0$ for all $i \leq n$ and $j \leq m$,

(iii) $\tau_i \not\trianglerighteq \tau_j^\frown 0$ for all $i, j \leq m$ with $i \neq j$.

By choice of $\langle c_\eta \rangle_{\eta \in 2^{<\omega}}$, there exists $\sigma'_0, ..., \sigma'_n, \tau'_0, ..., \tau'_m \in 2^{<\omega}$ such that

$$\langle \sigma'_0, ..., \sigma'_n, \tau'_0, ..., \tau'_m \rangle \sim_\alpha \langle \sigma_0, ..., \sigma_n, \tau_0, ..., \tau_m \rangle$$

and

$$b_{\sigma'_0} ... b_{\sigma'_n} b_{\tau'_0} ... b_{\tau'_m} \equiv_\Delta c_{\sigma_0} ... c_{\sigma_n} c_{\tau_0} ... c_{\tau_m},$$

where $\Delta = \{\psi_0, \psi_1\}$. By $\sim_\alpha$, the conditions (i), (ii), (iii) above hold on σ'_0, ..., σ'_n, τ'_0, ..., τ'_m. Let $X'\{\sigma'^\frown_0 0, ..., \sigma'^\frown_n 0, \tau'^\frown_0 1, ..., \tau'^\frown_m 1\}$. By (i), (ii), (iii), there are no $\eta', \nu' \in 2^{<\omega}$ such that $\eta'^\frown 1, \eta'^\frown 0^\frown \nu' \in X'$. Thus the set

$$\{\varphi(x, a_{\xi'}) : \xi' \in X'\}$$

is consistent since $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ witnesses SSOP_1 with φ. Thus

$$\{\psi_0(x, b_{\sigma'_0}), ..., \psi_0(x, b_{\sigma'_n})\} \cup \{\psi_1(x, b_{\tau'_0}), ..., \psi_1(x, b_{\tau'_m})\}$$

is consistent. Since $b_{\sigma'_0} ... b_{\sigma'_n} b_{\tau'_0} ... b_{\tau'_m}$ and $c_{\sigma_0} ... c_{\sigma_n} c_{\tau_0} ... c_{\tau_m}$ have the same Δ type,

$$\{\psi_0(x, c_{\sigma_0}), ..., \psi_0(x, c_{\sigma_n})\} \cup \{\psi_1(x, c_{\tau_0}), ..., \psi_1(x, c_{\tau_n})\}$$

is consistent as we desired. This proves claim. □

Thus we have a γ-indiscernible tree $\langle d_\eta \rangle_{\eta \in 2^{<\omega}}$ which witnesses SSOP_1 with φ.

Define a map $h : 2^{<\omega} \to 2^{<\omega}$ by

$$h(\eta) = \begin{cases} \langle\rangle & \text{if } \ \eta = \langle\rangle \\ h(\eta^-)^\frown\langle 01\rangle & \text{if } \ t(\eta) = 0 \\ h(\eta^-)^\frown\langle 1\rangle & \text{if } \ t(\eta) = 1, \end{cases}$$

and put $e_\eta = d_{h(\eta)}$ for each $\eta \in 2^{<\omega}$. As we observed in Theorem 2.11, $\langle e_\eta \rangle_{\eta \in 2^{<\omega}}$ is β-indiscernible. It is easy to show that for all $\eta, \nu \in 2^{<\omega}$,

$$(\eta \wedge \nu)^\frown 1 = \nu \ \text{ and } \ (\eta \wedge \nu)^\frown 0 \trianglelefteq \eta$$

if and only if

$$(h(\eta) \wedge h(\nu))^\frown 1 = h(\nu) \ \text{ and } \ (h(\eta) \wedge h(\nu))^\frown 0 \trianglelefteq h(\eta).$$

Therefore $\langle e_\eta \rangle_{\eta \in 2^{<\omega}}$ witnesses SSOP_1. □

Proposition 4.4. *If a theory T has an antichain tree $\langle \varphi(x,y), \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$, then there exists $\langle b_\eta \rangle_{\eta \in 2^{<\omega}}$ such that $\langle \varphi(x,y), \langle b_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ witnesses SSOP$_1$ modulo T.*

Proof. Suppose an antichain tree $\langle \varphi, \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ is given. Define a map $g : 2^{<\omega} \to 2^{<\omega}$ by

$$g(\eta) = \begin{cases} 0 & \text{if } \ \eta = \langle\rangle \\ g(\eta^-)^-{}^\frown\langle 1000\rangle & \text{if } \ t(\eta) = 0 \\ g(\eta^-)^-{}^\frown\langle 10\rangle & \text{if } \ t(\eta) = 1. \end{cases}$$

Let $b_\eta = a_{g(\eta)}$ for each $\eta \in 2^{<\omega}$. Then $\langle \varphi, \langle b_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ is a witness of SSOP_1. □

Now we are ready to modify the Claim 1.10.

Proposition 4.5. *Suppose that $\varphi(x, y)$ satisfies SOP$_1$, but there is no $n \in \omega$ such that the formula $\varphi_n(x, y_0, ..., y_{n-1}) = \bigwedge_{k<n} \varphi(x, y_k)$ satisfies SOP$_2$. Then there is a witness $\langle a_\eta : \eta \in 2^{<\omega} \rangle$ for $\varphi(x, y)$ satisfying SOP$_1$ which in addition satisfies*

(i) $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ witnesses SSOP$_1$ with φ,

(ii) $\langle a_\eta \rangle_{\eta \in 2^{<\omega}}$ is β-indiscernible.

Proof. By Theorem 3.8, there exists an antichain tree. By Proposition 4.4, there exists a witness of SSOP$_1$. And by Proposition 4.3, there exists a β-indiscernible witness of SSOP$_1$. □

In fact ATP and SSOP$_1$ are equivalent. The following proposition shows this along with Proposition 4.4.

Proposition 4.6. *If a theory T has a witness of SSOP$_1$, say $\langle \varphi, \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$, then φ also witnesses ATP modulo T.*

Proof. Suppose a witness of SSOP$_1$ $\langle \varphi, \langle a_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ is given. Define a map $g : 2^{<\omega} \to 2^{<\omega}$ by

$$g(\eta) = \begin{cases} 1 & \text{if } \eta = \langle\rangle \\ g(\eta^-)^\frown\langle 001 \rangle & \text{if } t(\eta) = 0 \\ g(\eta^-)^\frown\langle 011 \rangle & \text{if } t(\eta) = 1. \end{cases}$$

Let $b_\eta = a_{g(\eta)}$ for each $\eta \in 2^{<\omega}$. Then $\langle \varphi, \langle b_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ is an antichain tree. □

As we mentioned, it is obvious that having SSOP$_1$ implies having SOP$_1$. But the converse does not hold in general as follows.

Remark 4.7. Thus the class of SSOP_1 theories is a proper subclass of the class of SOP_1 theories.

Proof. To get a contradiction, let us assume that if T has SOP_1 then T has SSOP_1 for every theory T. Suppose T has SOP_2. Then T has SOP_1 and hence T has SSOP_1 by the assumption. By Proposition 4.6, T has ATP. By Proposition 3.5, T has TP_2. Therefore every SOP_2 theory has TP_2. But it is not true. □

Thus there exists a theory having SOP_1 but does not having SSOP_1. This theory is not ATP by Proposition 4.4, and hence the theory is SOP_2 by Corollary 3.9.

Chapter 5

An example of ATP theory having no quantifier free formula which witnesses SOP_2

In Chapter 3, we showed the existence of an antichain tree in SOP_1-NSOP_2 context. It is natural to ask if an antichain tree exists without classification theoretical hypothesis. We construct a structure of relational language whose theory has a quantifier free formula $\varphi(x, y)$ which forms an antichain tree, and every quantifier free formula in $\mathcal{L}$ does not witness SOP_2. In particular, $\bigwedge_{i<n} \varphi(x, y_i)$ do not witness SOP_2 for all $n \in \omega$. And the witness $\varphi(x, y)$ of ATP will also witness SOP_1 by Proposition 3.4. Therefore this theory witnesses difference of SOP_1 and SOP_2 at the level of formulas.

5.1 Construction

Definition 5.1. Let $n \leq \omega$. An antichain $X \subseteq 2^{<n}$ is called a maximal antichain if there is no antichain $Y \subseteq 2^{<n}$ such that $X \subsetneq Y$.

We begin the construction with language $\mathcal{L} = \{R\}$ where R is a binary relation symbol. For each $n \in \omega$, let $\alpha_n \in \omega$ be the number of all maximal antichains in $2^{<n}$,

and β_n be the set of all maximal antichains in $2^{<n}$. We can choose a bijection from α_n to β_n for each $n \in \omega$, say μ_n. For each $n \in \omega$, let A_n and B_n be finite sets such that $|A_n| = \alpha_n$ and $|B_n| = |2^{<n}|$. Their elements are denoted by

$$A_n = \{a^n_l : l < \alpha_n\}, \qquad B_n = \{b^n_\eta : \eta \in 2^{<n}\}.$$

Let C_n be the disjoint union of A_n and B_n for each $n \in \omega$. For each $n \in \omega$, let $\mathbb{C}_n$ be an $\mathcal{L}$-structure such that $\mathbb{C}_n = \langle C_n; R^{\mathbb{C}_n}\rangle$, where $R^{\mathbb{C}_n} = \{\langle a^n_l, b^n_\eta\rangle \in A_n \times B_n : \eta \in \mu_n(l)\}$.

For each $n \in \omega$, let ε_n be a map from $\alpha_n \cup 2^{<n}$ to $\alpha_{n+1} \cup 2^{<n+1}$ which maps $x \mapsto x$ for all $x \in \alpha_n \cup 2^{<n}$, and define $\varepsilon^*_n : C_n \to C_{n+1}$ by $a^n_l \mapsto a^{n+1}_{\varepsilon_n(l)}$ and $b^n_\eta \mapsto b^{n+1}_{\varepsilon_n(\eta)}$. Then ε^*_n is an embedding. Thus we can regard $\mathbb{C}_n$ as a substructure of $\mathbb{C}_{n+1}$ with respect to ε^*_n.

Let $\mathcal{P}_n$ be the power set of α_n, $\{\Gamma_i\}_{i<2^{\alpha_n}}$ an enumeration of $\mathcal{P}_n$, and p_n the cardinality of $\mathcal{P}_n$. For each $n \in \omega$, let δ_n be a map from $\alpha_n \cup 2^{<n}$ to $\alpha_{n+p_n} \cup 2^{<n+p_n}$ which maps $\eta \mapsto 0^{p_n}{}^\frown\eta$ for all $\eta \in 2^{<n}$ and $l \mapsto \delta_n(l)$ for all $l \in \alpha_n$ where

$$\delta_n(l) = \mu^{-1}_{n+p_n}\begin{pmatrix} (0^{p_n}{}^\frown\mu_n(l) \\ \cup\{0^{p_n-i-1}{}^\frown\langle 1\rangle : l \notin \Gamma_i\} \\ \cup\{0^{p_n-i-1}{}^\frown\langle 10\rangle : l \in \Gamma_i\} \\ \cup\{0^{p_n-i-1}{}^\frown\langle 11\rangle : l \in \Gamma_i\} \end{pmatrix}.$$

Define $\delta^*_n : C_n \to C_{n+p_n}$ by $a^n_l \mapsto a^{n+p_n}_{\delta_n(l)}$ and $b^n_\eta \mapsto b^{n+p_n}_{\delta_n(\eta)}$. Then δ^*_n is an embedding. Thus we can regard $\mathbb{C}_n$ as a substructure of $\mathbb{C}_{n+p_n}$ with respect to δ^*_n.

Then we obtain a chain of $\mathcal{L}$-structures,

$$\mathbb{C}_1 \xhookrightarrow{\varepsilon^*_1} \mathbb{C}_2 \xhookrightarrow{\delta^*_2} \mathbb{C}_{2+p_2} \xhookrightarrow{\varepsilon^*_{2+p_2}} \cdots \mathbb{C}_n \xhookrightarrow{\varepsilon^*_n} \mathbb{C}_{n+1} \xhookrightarrow{\delta^*_{n+1}} \mathbb{C}_{n+1+p_{n+1}} \cdots .$$

Thus we choose $\mathbb{C}'_n$ recursively as follows

$$\begin{aligned}
&\mathbb{C}'_1 = \mathbb{C}_1,\\
&\mathbb{C}'_2 = \mathbb{C}_2,\\
&\mathbb{C}'_{2i+1} = \mathbb{C}_{k+p_k} \text{ where } \mathbb{C}_k = \mathbb{C}'_{2i},\\
&\mathbb{C}'_{2i+2} = \mathbb{C}_{k+1} \text{ where } \mathbb{C}_k = \mathbb{C}'_{2i+1},
\end{aligned}$$

and for each $i \in \omega$, let $A'_i = A_k$ and $B'_i = B_k$ where $\mathbb{C}'_i = \mathbb{C}_k$. Let $\mathbb{C} = \bigcup_{i\in\omega} \mathbb{C}'_i$, $A = \bigcup_{i\in\omega} A'_i$, $B = \bigcup_{i\in\omega} B'_i$, and C be the universe of $\mathbb{C}$.

5.2 Verification

In this section we show that $\mathrm{Th}(\mathbb{C})$ has ATP and there is no quantifier free formula in $\mathcal{L}$ which has SOP_2 modulo $\mathrm{Th}(\mathbb{C})$. First we check some properties of $\mathrm{Th}(\mathbb{C})$. Recall that $\{\nu \in 2^{<\omega} : \nu \trianglelefteq \eta\}$ is linearly ordered by $\trianglelefteq$ for each η. This fact will be used frequently to prove the following propositions and remarks.

Remark 5.2. It is clear that either $\mathbb{C} \models \exists x R(x, c)$ or $\mathbb{C} \models \exists y R(c, y)$, for all $c \in C$. In fact $\mathbb{C} \models \exists x R(x, c)$ if and only if $c \in B$, and $\mathbb{C} \models \exists y R(c, y)$ if and only if $c \in A$.

Proposition 5.3. *$R(x, y)$ forms an antichain tree in $\mathrm{Th}(\mathbb{C})$.*

Proof. By compactness, it is enough to show that for any $n \in \omega$, there exists $\langle c_\eta \rangle_{\eta\in 2^{<n}}$ such that $\{R(x, c_\eta) : \eta \in X\}$ is consistent if and only if X is an antichain in $2^{<n}$. Fix $n \in \omega$, and let $c_\eta = b^k_\eta$ for each $\eta \in 2^{<n}$, where $\mathbb{C}_k = \mathbb{C}'_n$. Note that $k \geq n$.

First we show that if $X \subseteq 2^{<n}$ is an antichain, then $\{R(x, c_\eta) : \eta \in X\}$ is consistent. Since $2^{<n} \subseteq 2^{<k}$, X can be regarded as an antichain in $2^{<k}$. Hence, there exists a maximal antichain $\bar{X} \subseteq 2^{<k}$ containing X. Let $l = \mu_k^{-1}(\bar{X})$. Then $\langle a^k_l, c_\eta\rangle \in R^{\mathbb{C}_k}$ for all $\eta \in X$ since $\langle a^k_l, c_\eta\rangle = \langle a^k_l, b^k_\eta\rangle$ and $\eta \in \bar{X} = \mu_k(\mu_k^{-1}(\bar{X})) = \mu_k(l)$ for all $\eta \in X$. Thus $\{R(x, c_\eta) : \eta \in X\}$ is consistent.

For the converse, suppose $X \subseteq 2^{<n}$ is not an antichain. Then there exist $\eta, \nu \in X$ such that $\eta \not\perp \nu$ in $2^{<n}$. Thus $\eta \not\perp \nu$ in $2^{<k}$. If $\{R(x, c_\eta) : \eta \in X\}$ is consistent, then $\{R(x, b^k_\eta), R(x, b^k_\nu)\}$ is consistent. Hence there exists $a \in A$ such that $\mathbb{C} \models R(a, b^k_\eta) \wedge R(a, b^k_\nu)$. By the construction of $\mathbb{C}$, there exists $k' \in \omega$ such that $a \in A_{k'}$ and we may assume $k = k'$. Hence there exists $l < \alpha_k$ such that $\mathbb{C} \models R(a^k_l, b^k_\eta) \wedge R(a^k_l, b^k_\nu)$. Thus $\eta, \nu \in \mu_k(l)$. There exists an antichain in $2^{<k}$ containing η and ν but it is not possible since $\eta \not\perp \nu$. □

Proposition 5.4. *There is no $b_0, b_1, b_2, b_3 \in B$ such that*

$$\begin{aligned}\mathbb{C} \models \quad & b_0 \neq b_1 \wedge b_2 \neq b_3 \\ & \wedge \exists x(R(x, b_0) \wedge R(x, b_1)) \wedge \exists x(R(x, b_2) \wedge R(x, b_3)) \\ & \wedge \neg\exists x(R(x, b_0) \wedge R(x, b_2)) \wedge \neg\exists x(R(x, b_0) \wedge R(x, b_3)) \\ & \wedge \neg\exists x(R(x, b_1) \wedge R(x, b_2)) \wedge \neg\exists x(R(x, b_1) \wedge R(x, b_3)).\end{aligned}$$

Proof. Suppose such b_0, b_1, b_2, b_3 exist. We may assume they are $b^n_{\eta_0}, b^n_{\eta_1}, b^n_{\eta_2}, b^n_{\eta_3}$ for some $n \in \omega$ and $\eta_0, \eta_1, \eta_2, \eta_3 \in 2^{<n}$. Since $\{R(x, b^n_{\eta_0}),\ R(x, b^n_{\eta_1})\}$ and $\{R(x, b^n_{\eta_2}),\ R(x, b^n_{\eta_3})\}$ are consistent, $b^n_{\eta_0} \perp b^n_{\eta_1}$ and $b^n_{\eta_2} \perp b^n_{\eta_3}$ in $2^{<n}$. Since $\{R(x, b^n_{\eta_0}), R(x, b^n_{\eta_2})\}$, $\{R(x, b^n_{\eta_0}), R(x, b^n_{\eta_3})\}$, $\{R(x, b^n_{\eta_1}), R(x, b^n_{\eta_2})\}$, $\{R(x, b^n_{\eta_1}), R(x, b^n_{\eta_3})\}$ are inconsistent, $b^n_{\eta_0} \not\perp b^n_{\eta_2}$, $b^n_{\eta_0} \not\perp b^n_{\eta_3}$, $b^n_{\eta_1} \not\perp b^n_{\eta_2}$, $b^n_{\eta_1} \not\perp b^n_{\eta_3}$. Thus $b^n_{\eta_2}, b^n_{\eta_3} \trianglelefteq b^n_{\eta_0} \wedge b^n_{\eta_1}$. Since $\{\nu \in 2^{<n} : \nu \trianglelefteq \nu' \wedge \nu''\}$ is linearly ordered by $\trianglelefteq$ for all $\nu' \neq \nu''$, we have $b^n_{\eta_2} \not\perp b^n_{\eta_3}$ and it is a contradiction. □

Proposition 5.5. *For all $b_0, ..., b_n \in B$, if $\mathbb{C} \models \neg\exists x(R(x, b_0) \wedge \cdots \wedge R(x, b_n))$, then there exist $i < j \leq n$ such that $\mathbb{C} \models \neg\exists x(R(x, b_i) \wedge R(x, b_j))$.*

Proof. Suppose $b_0, ..., b_n \in B$ and $\mathbb{C} \models \exists x(R(x, b_i) \wedge R(x, b_j))$ for all $i, j \leq n$. We may assume that there exists $k \in \omega$ such that for each $i < n$, there exists $\eta_i \in 2^{<k}$

such that $b_i = b^k_{\eta_i}$. As we observed in proof of Proposition 5.3, $\langle R(x,y), \langle b^k_\eta \rangle_{\eta \in 2^{<k}} \rangle$ forms an antichain tree with height k. Since $\{R(x, b^k_{\eta_i}), R(x, b^k_{\eta_j})\}$ is consistent for each $i, j < n$, the set of indices $\{\eta_0, ..., \eta_n\}$ is pairwisely incomparable. Thus $\{\eta_0, ..., \eta_n\}$ is an antichain in $2^{<k}$. Thus $\{(R(x, b^k_{\eta_0}), ..., R(x, b^k_{\eta_{n-1}})\}$ is consistent in $\mathbb{C}_k$, and hence it is consistent in $\mathbb{C}$. Therefore $\mathbb{C} \models \exists x(R(x, b_0) \wedge \cdots \wedge R(x, b_n))$. □

Proposition 5.6. *For all $a_0, ..., a_n, a'_0, ..., a'_m \in A$, if $a_i \neq a'_j$ for each $i \leq n$, $j \leq m$, then $\mathbb{C} \models \exists y(R(a_0, y) \wedge \cdots \wedge R(a_n, y) \wedge \neg R(a'_0, y) \wedge \cdots \wedge \neg R(a'_m, y))$.*

Proof. We may assume $a_0, ..., a_n, a'_0, ..., a'_m \in A_k$ for some $k \in \omega$, where $\mathbb{C}_k = \mathbb{C}'_{2d}$ for some $i \in \omega$. Hence there are $l_0, ..., l_n, l'_0, ..., l'_m < \alpha_k$ such that $a_0 = a^k_{l_0}, \ldots, a_n = a^k_{l_n}$ $a'_0 = a^k_{l'_0}$, ... , $a'_m = a^k_{l'_m}$ and $l_i \neq l'_j$ for each i and j. Let $\{\Gamma_i\}_{i<2^{\alpha_k}}$ be the enumeration of $\mathcal{P}_k$, the power set of α_k. Then there exists e such that $\Gamma_e = \{l'_0, ..., l'_m\}$. By the construction of δ_k, the maximal antichain $\mu_{k+p_k}(\delta_k(l'_j))$ in β_{k+p_k} does not contain $0^{(p_k-e-1)\frown}\langle 1\rangle$ for all j. On the other hand, $\mu_{k+p_k}(\delta_k(l_i))$ contains $0^{p_k-e-1\frown}\langle 1\rangle$ for all i since $l_i \notin \Gamma_e$. Thus we have

$$\mathbb{C} \models R(a^{k+p_k}_{\delta_k(l_i)}, b^{k+p_k}_{0^{p_k-e-1\frown}\langle 1\rangle})$$

for all i, and

$$\mathbb{C} \models \neg R(a^{k+p_k}_{\delta_k(l'_j)}, b^{k+p_k}_{0^{p_k-e-1\frown}\langle 1\rangle})$$

for all j. Since δ^*_k is an embedding from $\mathbb{C}'_{2d} = \mathbb{C}_k$ to $\mathbb{C}'_{2d+1} = \mathbb{C}_{k+p_k}$, we can say that there is $b \in B$ such that $\mathbb{C} \models R(a^k_{l_i}, b)$ for all i, and $\mathbb{C} \models \neg R(a^k_{l'_j}, b)$ for all j. This completes the proof. □

Proposition 5.7. *For all* $b_0, ..., b_n, b'_0, ..., b'_m \in B$, *if* $b_i \neq b'_j$ *for each* $i \leq n, j \leq m$, *and* $\mathbb{C} \models \exists x(R(x, b_i) \wedge R(x, b_j))$ *for all* i, j, *then* $\mathbb{C} \models \exists x(R(x, b_0) \wedge \cdots \wedge R(x, b_n) \wedge \neg R(x, b'_0) \wedge \cdots \wedge \neg R(x, b'_m))$.

Proof. As in Proposition 5.6, we may assume $b_0, ..., b_n, b'_0, ...b'_m \in B_k = B'_{2d+1}$ for some $k, d \in \omega$, hence there are $\eta_0, ..., \eta_n, \eta'_0, ..., \eta'_m \in 2^{<\kappa}$ such that $b_0 = b^k_{\eta_0}, ..., b_n = b^k_{\eta_n}, b'_0 = b^k_{\eta'_0}, ..., b'_0 = b^k_{\eta'_m}$ and $\eta_i \neq \eta'_j$ for each $i \leq n$ and $j \leq m$. Since $\mathbb{C} \models \exists x(R(x, b_{\eta_i}) \wedge R(x, b_{\eta_j}))$ for each $i, j \leq n$, the set $\{\eta_0, ..., \eta_n\}$ forms an antichain in $2^{<k}$. By the construction we may assume $\mathbb{C}_k = \mathbb{C}'_{2d+1}$ is a substructure of $\mathbb{C}_{k+1} = \mathbb{C}'_{2d+2}$ with respect to ε^*_k. Thus $\eta_0, ..., \eta_n, \eta'_0, ..., \eta'_m$ can be regarded as elements of $2^{<k+1}$. Now we choose an antichain tree in $2^{<k+1}$ which contains $\eta_0, ..., \eta_n$ and does not contain $\eta'_0, ...\eta'_m$. Let Λ be a maximal antichain in $2^{<k+1}$ which is defined by

$$\{\nu \in 2^{<k+1} : \nu \in 2^k \text{ and } \eta_i \not\trianglelefteq \nu \text{ for all } i \leq n\} \cup \{\eta_0, ..., \eta_n\}.$$

Then $\eta_i \in \Lambda$ for all $i \leq n$ and $\eta'_j \notin \Lambda$ for all $j \leq m$. Let $l = \mu^-_{k+1}(\Lambda)$. Then $\mathbb{C} \models R(a^{k+1}_l, b^{k+1}_{\eta_i})$ for all $i \leq n$, and $\mathbb{C} \models \neg R(a^{k+1}_l, b^{k+1}_{\eta'_j})$ for all $j \leq m$. This completes the proof. □

Now we are ready to prove that Th($\mathbb{C}$) has no quantifier free formula which witnesses SOP_2. We begin the proof by showing that some conjunctions of basic formulas do not witness SOP_2.

Proposition 5.8. *The formula* $\varphi(\bar{x}, \bar{x}', y) = \bigwedge_{i \leq n} R(x_i, y) \wedge \bigwedge_{j \leq m} \neg R(x'_j, y)$ *does not witness SOP*$_2$ *modulo Th*($\mathbb{C}$) *for all* $n, m \in \omega$. *That is, there is no* $\langle \bar{c}^\eta, \bar{c}'^\eta \rangle_{\eta \in 2^{<\omega}} = \langle c^\eta_0, ..., c^\eta_n, c'^\eta_0, ..., c'^\eta_m \rangle_{\eta \in 2^{<\omega}}$ *in a monster model* $\mathbb{M}$ *of Th*($\mathbb{C}$), *such that*

(i) $\{\varphi(\bar{c}^{\eta \restriction n}, \bar{c}'^{\eta \restriction n}, y) : n \in \omega\}$ *is consistent for all* $\eta \in 2^\omega$

(ii) $\{\varphi(\bar{c}^{\eta}, \bar{c}^{'\eta}, y), \varphi(\bar{c}^{\nu}, \bar{c}^{'\nu}, y)\}$ *is inconsistent for all* $\eta \perp \nu$ *in* $2^{<\omega}$.

Proof. To get a contradiction we assume there exists $\langle \bar{c}^{\eta}, \bar{c}^{'\eta} \rangle_{\eta \in 2^{<\omega}}$ which witnesses SOP_2 with $\varphi(\bar{x}, \bar{x}', y)$. We may assume $\langle \bar{c}_{\eta} \rangle_{\eta \in 2^{<\omega}}$ is strongly indiscernible by the modeling property. Thus

$$\{R(c_0^{\langle 0 \rangle}, y), ..., R(c_n^{\langle 0 \rangle}, y), \neg R(c_0^{'\langle 0 \rangle}, y), ..., \neg R(c_m^{'\langle 0 \rangle}, y),$$
$$R(c_0^{\langle 1 \rangle}, y), ..., R(c_n^{\langle 1 \rangle}, y), \neg R(c_0^{'\langle 1 \rangle}, y), ..., \neg R(c_m^{'\langle 1 \rangle}, y)\}$$

is inconsistent. By Proposition 5.6, there are $i \leq n$ and $j \leq m$ such that $c_i^{\langle 0 \rangle} = c_j^{'\langle 1 \rangle}$ or $c_i^{\langle 1 \rangle} = c_j^{'\langle 0 \rangle}$. We may assume $c_i^{\langle 0 \rangle} = c_j^{'\langle 1 \rangle}$. Then $c_i^{\langle 0 \rangle} = c_j^{'\langle 10 \rangle}$, $c_i^{\langle 0 \rangle} = c_j^{'\langle 11 \rangle}$ and $c_i^{\langle 10 \rangle} = c_j^{'\langle 11 \rangle}$ by strong indiscernibility. Thus $c_i^{\langle 10 \rangle} = c_j^{'\langle 11 \rangle} = c_i^{\langle 0 \rangle} = c_j^{'\langle 10 \rangle}$. But it is a contradiction because the set $\{R(c_0^{\langle 10 \rangle}, y), ..., R(c_n^{\langle 10 \rangle}, y), \neg R(c_0^{'\langle 10 \rangle}, y), ..., \neg R(c_m^{'\langle 10 \rangle}, y)\}$ is consistent. □

Proposition 5.9. *The formula* $\varphi(x, \bar{y}, \bar{y}') = \bigwedge_{i \leq n} R(x, y_i) \wedge \bigwedge_{j \leq m} \neg R(x, y'_j)$ *does not witness SOP*$_2$ *modulo Th*$(\mathbb{C})$ *for all* $n, m \in \omega$. *That is, there is no* $\langle \bar{c}^{\eta}, \bar{c}^{'\eta} \rangle_{\eta \in 2^{<\omega}} = \langle c_0^{\eta}, ..., c_n^{\eta}, c_0^{'\eta}, ..., c_m^{'\eta} \rangle_{\eta \in 2^{<\omega}}$ *in a monster model* $\mathbb{M}$ *of Th*$(\mathbb{C})$, *such that*

(i) $\{\varphi(x, \bar{c}^{\eta \restriction n}, \bar{c}^{'\eta \restriction n}) : n \in \omega\}$ *is consistent for all* $\eta \in 2^{\omega}$

(ii) $\{\varphi(x, \bar{c}^{\eta}, \bar{c}^{'\eta}), \varphi(x, \bar{c}^{\nu}, \bar{c}^{'\nu})\}$ *is inconsistent for all* $\eta \perp \nu$ *in* $2^{<\omega}$.

Proof. As in Proposition 5.8, we assume there is a strongly indiscernible tree of parameters $\langle \bar{c}^{\eta}, \bar{c}^{'\eta} \rangle_{\eta \in 2^{<\omega}}$ which witnesses SOP_2 with $\varphi(x, \bar{y}, \bar{y}')$. Then

$$\{R(x, c_0^{\langle 0 \rangle}), ..., R(x, c_n^{\langle 0 \rangle}), \neg R(x, c_0^{'\langle 0 \rangle}), ..., \neg R(x, c_m^{'\langle 0 \rangle}),$$
$$R(x, c_0^{\langle 1 \rangle}), ..., R(x, c_n^{\langle 1 \rangle}), \neg R(x, c_0^{'\langle 1 \rangle}), ..., \neg R(x, c_m^{'\langle 1 \rangle})\}$$

is inconsistent. By Proposition 5.7, there are $i \leq n$, $j \leq m$ such that $c_i^{\langle 0 \rangle} = c_j^{'\langle 1 \rangle} \vee c_i^{\langle 1 \rangle} = c_j^{'\langle 0 \rangle}$, or there are $i, j \leq n$ such that $\{R(x, c_i^{\langle 0 \rangle}), R(x, c_j^{\langle 1 \rangle})\}$ is inconsistent.

If it is in the first case, then we can find a contradiction as we observed in Proposition 5.8. Thus we assume $\{R(x, c_i^{\langle 0\rangle}), R(x, c_j^{\langle 1\rangle})\}$ is inconsistent for some $i, j \leq n$. By strong indiscernibility, $\{R(x, c_i^{\langle 0\rangle}), R(x, c_j^{\langle 11\rangle})\}$, $\{R(x, c_i^{\langle 00\rangle}), R(x, c_j^{\langle 1\rangle})\}$, and $\{R(x, c_i^{\langle 00\rangle}), R(x, c_j^{\langle 11\rangle})\}$ are inconsistent. On the other hand, $\{R(x, c_i^{\langle 0\rangle}), R(x, c_i^{\langle 00\rangle})\}$ and $\{R(x, c_j^{\langle 1\rangle}), R(x, c_j^{\langle 11\rangle})\}$ are consistent by the path consistency condition. But then $c_i^{\langle 0\rangle}, c_i^{\langle 00\rangle}, c_j^{\langle 1\rangle}, c_j^{\langle 11\rangle}$ violate Proposition 5.4. This completes the proof. □

To generalize the result of Propositions 5.9 and 5.8 to the case of quantifier free formulas, we need the following fact and propositions.

Fact 5.10. [3, Corollary 4.11] *Let T be an arbitrary theory. If T has SOP$_2$ then there is a formula in a single free variable witnessing SOP$_2$.*

Proposition 5.11. *Let T be a theory and suppose $\varphi(x, \bar{y}) \vee \psi(x, \bar{z})$ witnesses SOP$_2$ modulo T, where x is a free variable, and $\bar{y}$, $\bar{z}$ are parameter variables. Then $\varphi(x, \bar{y})$ witnesses SOP$_2$ or $\psi(x, \bar{z})$ witnesses SOP$_2$.*

Proof. Let $\langle \bar{b}^\eta, \bar{c}^\eta \rangle_{\eta \in 2^{<\omega}} \subseteq \mathbb{M}$ witnesses SOP$_2$ with $\varphi \vee \psi$, where $\mathbb{M}$ is a monster model of T. We may assume the tree of parameters is strongly indiscernible. Since $\{\varphi(x, \bar{b}^\eta) \vee \psi(x, \bar{c}^\eta), \varphi(x, \bar{b}^\nu) \vee \psi(x, \bar{c}^\nu)\}$ is inconsistent for all $\eta \perp \nu$, it is clear that $\{\varphi(x, \bar{b}^\eta), \varphi(x, \bar{b}^\nu)\}$ and $\{\psi(x, \bar{c}^\eta), \psi(x, \bar{c}^\nu)\}$ are inconsistent for all $\eta \perp \nu$. Since $\{\varphi(x, \bar{b}^{\langle 0^n\rangle}) \vee \psi(x, \bar{c}^{\langle 0^n\rangle}) : n \in \omega\}$ is consistent, there exists $a \in \mathbb{M}$ such that $\mathbb{M} \models \varphi(a, \bar{b}^{\langle 0^n\rangle}) \vee \psi(a, \bar{c}^{\langle 0^n\rangle})$ for all $n \in \omega$. By the pigeon hole principle, there exists an infinite subset I of ω, such that $\mathbb{M} \models \varphi(a, \bar{b}^{\langle 0^i\rangle})$ for all $i \in I$, or $\mathbb{M} \models \psi(a, \bar{c}^{\langle 0^i\rangle})$ for all $i \in I$. Without loss of generality, we may assume $\mathbb{M} \models \varphi(a, \bar{b}^{\langle 0^i\rangle})$ for all $i \in I$. By strong indiscernibility, $\{\varphi(x, \bar{b}^{\eta\restriction n}) : n \in \omega\}$ is consistent for all $\eta \in 2^\omega$ because

$\langle \eta \restriction n \rangle_{n\in\omega} \sim_{str} \langle 0^i \rangle_{i\in I}$. Thus $\varphi(x, \bar{y})$ witnesses SOP_2 with $\langle \bar{b}^\eta \rangle_{\eta\in 2^{<\omega}}$. □

Proposition 5.12. *In Th($\mathbb{C}$), there is no quantifier free formula witnessing SOP$_2$.*

Proof. Suppose φ is a quantifier free formula witnessing SOP_2 modulo Th($\mathbb{C}$). By Proposition 5.10, we may assume φ is of the form $\varphi(z, \bar{w})$ where z is a single free variable, $\bar{w}$ is a parameter variable. By using disjunctive normal form and Proposition 5.11, we may assume φ is a conjunction of basic formulas. Thus we may assume φ is of the form

$$\bigwedge R(z, w_i) \wedge \bigwedge \neg R(z, w'_j) \wedge \bigwedge R(w''_k, z) \wedge \bigwedge \neg R(w'''_l, z).$$

But by the construction of $\mathbb{C}$, we have either $\mathbb{C} \models \exists y R(c, y)$ or $\mathbb{C} \models \exists x R(x, c)$ for all $c \in \mathbb{C}$. Hence φ is of the form either

$$\bigwedge R(z, w_i) \wedge \bigwedge \neg R(z, w'_j)$$

or

$$\bigwedge R(w_i, z) \wedge \bigwedge \neg R(w'_j, z).$$

By Propositions 5.8 and 5.9 we know that both of the above formulas can not witness SOP_2 modulo Th($\mathbb{C}$). Thus we have a contradiction and this completes the proof. □

Corollary 5.13. *SOP$_1$ and SOP$_2$ are distinct at the level of formulas.*

Proof. By Proposition 5.12, there is no quantifier free formula witnessing SOP_2 in Th($\mathbb{C}$). In particular, $\bigwedge_{i\leq n} R(x, y_i)$ does not witness SOP_2 modulo Th($\mathbb{C}$) for all $n \in \omega$. By Proposition 5.3 and Proposition 3.4, the formula $R(x, y)$ witnesses SOP_1. Hence SOP_1 and SOP_2 are distinct at the level of formulas. □

Remark 5.14. But $\mathrm{Th}(\mathbb{C})$ has a witness of SOP_2. Define $\varphi(x, y)$ by

$$\begin{aligned}
&x \neq y \\
&\wedge \neg \exists w(R(w, x) \wedge R(w, y)) \\
&\wedge \exists z(\exists w(R(w, x) \wedge R(w, z)) \wedge \neg \exists w(R(w, y) \wedge R(w, z))).
\end{aligned}$$

Then φ says "y is a predecessor of x in the set of parameters" (*i.e.*, $y \trianglelefteq x$). For each $\eta \in 2^{<\omega}$, let $b_\eta = b^n_\eta$ for some $n \in \omega$. By the constructions of $\mathbb{C}$, b_η is well-defined. Then $\{\varphi(x, b_\eta), \varphi(x, b_\nu)\}$ is inconsistent whenever $\eta \perp \nu$. By compactness $\{\varphi(x, b_{\eta\restriction n}) : n < \omega\}$ is consistent for each $\eta \in {}^{\omega}2$. Thus $\langle \varphi(x, y), \langle b_\eta \rangle_{\eta \in 2^{<\omega}} \rangle$ witnesses SOP_2 modulo $\mathrm{Th}(\mathbb{C})$.

Chapter 6

Antichain collapse lemmas

In this chapter we show that if a theory has ATP, then there exists a formula in a single free variable which witnesses ATP. As a corollary, SSOP_1 is also witnessed by a formula in a single free variable. Beginning to study ATP, it is natural to ask whether ATP is witnessed in a single free variable as other tree properties are. Following the notation of [12], let us call this problem one-variable theorem for ATP. The one-variable theorems for SOP_1 and TP_2 are solved by Ramsey in [12], and by Chernikov in [2] respectively. Chernikov and Ramsey also solved the one-variable theorem for SOP_2 in [3, Corollary 4.11] by using "Path-Collapse Lemma" which says

Fact 6.1. [3, Lemma 4.6] (Path Collapse) *Suppose κ is an infinite cardinal, $\langle a_\eta \rangle_{\eta \in 2^{<\kappa}}$ is a tree str-indiscernible over a set of parameters C and, moreover, $\langle a_{0^\alpha} \rangle_{0<\alpha<\omega}$ is order indiscernible over cC. Let*

$$p(y; \bar{z}) = tp(c; \langle a_{0^\frown 0^\gamma} \rangle_{\gamma<\kappa}/C).$$

Then if

$$p(y : \langle a_{0^\frown 0^\gamma} \rangle_{\gamma<\kappa}) \cup p(y : \langle a_{1^\frown 0^\gamma} \rangle_{\gamma<\kappa})$$

is not consistent, then T has SOP_2, witnessed by a formula with free variables y.

To obtain a witness of ATP in a single free variable, we find an appropriate statement which is similar to the path-collapse lemma. In short, we find the condition for the consistency of $p(y;\bar{a}) \cup p(y;\bar{b})$ where $\bar{a}$, $\bar{b}$ are antichains, $p(y;\bar{z}) = \mathrm{tp}(c;\bar{a})$, and $\bar{a}$ is str-indiscernible over c. To do this, we first develop some techniques to control homogeneity of antichains.

6.1 Universal antichains and δ-indiscernibility

Notation 6.2. Let α, β be ordinals and suppose $\eta \in \alpha^{<\beta}$, $X \subseteq \alpha^{<\beta}$. By $\eta^\frown X$ and $X^\frown \eta$, we mean $\{\eta^\frown x : x \in X\}$ and $\{x^\frown \eta : x \in X\}$ respectively.

The following definitions and notations are from [13], [6] and [10].

Definition 6.3. Let $\mathcal{L}$ be a language.

(i) Let M be an $\mathcal{L}$-structure. By $age(M)$, we mean the class of all finitely generated $\mathcal{L}$-structures which is isormophic to a substructure of M.

(ii) Let A and C be $\mathcal{L}$-structures. By *A-substruture of C*, we mean the class of $\mathcal{L}$-structure $A' \subseteq C$ isomorphic to A. We write it $\binom{C}{A}$.

(iii) For each $k \in \omega \backslash \{0\}$, we call a map from $\binom{C}{A}$ to k a *k-coloring of* $\binom{C}{A}$.

(iv) Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures, $A, B, C \in \mathcal{K}$, and $k \in \omega \backslash \{0\}$. By $C \to (B)^A_k$, we mean that for all k-colorings f of $\binom{C}{A}$, there exists $B' \subseteq C$ such that B' is $\mathcal{L}$-isomorphic to B and $f \restriction \binom{B'}{A}$ is constant.

(v) A class $\mathcal{K}$ of $\mathcal{L}$-structures is called a *Ramsey class* if for any $A, B \in \mathcal{K}$ and $k \in \omega \backslash \{0\}$, there exists $C \in \mathcal{K}$ such that $C \to (B)^A_k$.

(vi) An $\mathcal{L}$-structure $\mathcal{I}$ is said to be *locally finite* if for all finite subset X of $\mathcal{I}$, there is a finite substructure I_0 of $\mathcal{I}$ containing X.

(vii) Let us recall the definition of indiscernibility and modeling property in Section 2.1. They can be defined in a more general way as follows. Let $\mathcal{L}$ and $\mathcal{L}'$ be languages, T be an $\mathcal{L}$-theory, $\mathbb{M}$ be a monster model of T. For an $\mathcal{L}'$-structure $\mathcal{I}$, we say an $\mathcal{I}$-indexed set $\langle a_i \rangle_{i \in \mathcal{I}} \subseteq \mathbb{M}$ is $\mathcal{L}'$-indiscernible if $\bar{a}_{\bar{i}} \equiv \bar{a}_{\bar{j}}$ for all $\bar{i}, \bar{j} \subseteq \mathcal{I}$ whenever the quantifier free $\mathcal{L}'$-types of $\bar{i}$ and $\bar{j}$ are the same. We say $\mathcal{I}$-indexed set $\langle b_i \rangle_{i \in \mathcal{I}} \subseteq \mathbb{M}$ is $\mathcal{L}'$-based on $\langle a_i \rangle_{i \in \mathcal{I}} \subseteq \mathbb{M}$ if for any finite set Φ of $\mathcal{L}$-formulas and $\bar{i} \in \mathcal{I}$, there exists $\bar{j} \in \mathcal{I}$ such that $\bar{a}_{\bar{i}} \equiv_\Phi \bar{a}_{\bar{j}}$ and the quantifier free $\mathcal{L}'$-types of $\bar{i}$ and $\bar{j}$ are the same. We say $\mathcal{I}$-indexed sets have the modeling property with respect to $\mathcal{L}'$ (sometimes we write it $\mathcal{L}'$-modeling property) if for any $\langle a_i \rangle_{i \in \mathcal{I}} \subseteq \mathbb{M}$, there is an $\mathcal{L}'$-indiscernible $\langle b_i \rangle_{i \in \mathcal{I}} \subseteq \mathbb{M}$ which is $\mathcal{L}'$-based on $\langle a_i \rangle_{i \in \mathcal{I}}$.

The following fact proved by Scow in [13], plays an important role in this section. We note that the statement quoted below is a part of the original theorem.

Fact 6.4. [13, Theorem 3.12] *Let $\mathcal{I}$ be a locally finite $\mathcal{L}$-structure, where $\mathcal{L}$ is a finite language, then $\mathcal{I}$-indexed indiscernible sets have the modeling property if and only if age$(\mathcal{I})$ is a Ramsey class.*

In the rest of this section we find the appropriate language which describes the relation between elements of an antichain, and show that some specific antichains have the modeling property with respect to the language. And by using this we can show that for a given antichain tree $\langle \varphi(x, y), \langle a_\eta \rangle \rangle$, we can find a realization c of the set of formula $\{\varphi(x, a_\eta)\}_{\eta \in X}$ where X is an antichain such that $\langle a_\eta \rangle_{\eta \in X}$ is str-indiscernible over c.

Definition 6.5. Let I be a linearly ordered set, α be an ordinal and $X \subseteq I^{<\alpha}$ be an antichain. We say X is *universal* if for each finite antichain Y in $I^{<\alpha}$, there exists $X_0 \subseteq X$ such that $Y \sim_{str} X_0$. We say X is *dense* if for $x_0, x_1 \in X$ with $x_0 <_{lex} x_1$, there exist $y_0, y_{0.5}, y_1 \in X$ such that $x_0 \wedge y_0 \rhd y_0 \wedge x_1$, $x_0 \wedge y_{0.5} = y_{0.5} \wedge x_1$, $x_0 \wedge y_1 \lhd y_1 \wedge x_1$, and $x_0 <_{lex} y_0 <_{lex} y_{0.5} <_{lex} y_1 <_{lex} x_1$. We say X is *endless* if for all $\eta \in X$, there exists $\nu, \xi \in X$ such that $\nu <_{lex} \eta <_{lex} \xi$.

Examples 6.6. If λ is a limit ordinal, then $\mathbb{Q}^\lambda$ is an endless dense universal antichain. $\omega^\lambda, 2^\lambda$ are also universal, but they are not endless. ω^λ is dense but 2^λ is not dense.

We can construct universal antichain without using limit ordinals. In $2^{<\omega}$, $\bar{X} = \bigcup_{i\in\omega}(1^i{}^\frown 0{}^\frown X_i)$ is also an universal antichain where $\{X_i\}_{i\in\omega}$ is an enumeration of all finite antichain in $2^{<\omega}$. But this universal antichain is not dense, not endless.

Notation 6.7. Let $\mathcal{L}_\delta = \{\Delta, <_{lex}\}$ be a language where Δ is a quaternary relation symbol.

$\omega^{\leq\omega}$ can be regarded as an $\mathcal{L}_{str}$-structure as usual. For each $X \subseteq \omega^{\leq\omega}$, if X is an antichain, then we can regard X as an $\mathcal{L}_\delta$-structure by interpreting Δ by

$$\Delta(\eta_0, \eta_1, \eta_2, \eta_3) \text{ if and only if } \eta_0 \wedge \eta_1 \trianglelefteq \eta_2 \wedge \eta_3,$$

where $\trianglelefteq$ and $\wedge$ are as usual in $\omega^{\leq\omega}$. From now, by δ-indiscernibility, δ-basedness, and δ-modeling property, we mean the $\mathcal{L}_\delta$-indiscernibility, the $\mathcal{L}_\delta$-basedness, and the $\mathcal{L}_\delta$-modeling property respectively. For L_δ-structure X and $\bar{\eta}, \bar{\nu} \subseteq X$, by $\bar{\eta} \sim_\delta \bar{\nu}$ we mean that the quantifier free $\mathcal{L}_\delta$-types of $\bar{\eta}$ and $\bar{\nu}$ are the same.

Lemma 6.8. *Let $\kappa > \omega$, and $\bar{\eta} = \langle \eta_0, ..., \eta_n \rangle$ and $\bar{\nu} = \langle \nu_0, ..., \nu_n \rangle$ be tuples in ω^ω. If $\bar{\eta} \sim_\delta \bar{\nu}$, then $\mathrm{cl}(\bar{\eta}) \sim_{str} \mathrm{cl}(\bar{\nu})$ in $\omega^{<\kappa}$. Conversely, if $\mathrm{cl}(\bar{\eta}) \sim_{str} \mathrm{cl}(\bar{\nu})$ in $\omega^{<\kappa}$, then*

$\bar{\eta} \sim_\delta \bar{\nu}$. *Therefore, if* $\langle a_\eta \rangle_{\eta \in \omega^{<\kappa}}$ *is str-indiscernible, then* $\langle a_\eta \rangle_{\eta \in \omega^\omega}$ *is* δ*-indiscernible.*

Proof. Suppose $\bar{\eta} \sim_\delta \bar{\nu}$. To show $\mathrm{cl}(\bar{\eta}) \sim_{str} \mathrm{cl}(\bar{\nu})$, it is enough to show that

$$\eta_i \wedge \eta_j \trianglelefteq \eta_k \wedge \eta_l \text{ if and only if } \nu_i \wedge \nu_j \trianglelefteq \nu_k \wedge \nu_l,$$

$$\eta_i \wedge \eta_j <_{lex} \eta_k \wedge \eta_l \text{ if and only if } \nu_i \wedge \nu_j <_{lex} \nu_k \wedge \nu_l,$$

for all $i, j, k, l \leq n$. The first condition is given by Δ. Thus it is enough to show that the second condition holds. Suppose $\eta_i \wedge \eta_j <_{lex} \eta_k \wedge \eta_l$. If $\eta_i \wedge \eta_j \trianglelefteq \eta_k \wedge \eta_l$, then by Δ, we have $\nu_i \wedge \nu_j \trianglelefteq \nu_k \wedge \nu_l$ and hence $\nu_i \wedge \nu_j <_{lex} \nu_k \wedge \nu_l$. If $\eta_i \wedge \eta_j \not\trianglelefteq \eta_k \wedge \eta_l$, then $\eta_i \wedge \eta_j \not\trianglerighteq \eta_k \wedge \eta_l$ and $\eta_i, \eta_j <_{lex} \eta_k, \eta_l$ thus we have $\nu_i \wedge \nu_j \not\trianglelefteq \nu_k \wedge \nu_l$, $\nu_i \wedge \nu_j \not\trianglerighteq \nu_k \wedge \nu_l$, and $\nu_i, \nu_j <_{lex} \nu_k, \nu_l$. Therefore $\nu_i \wedge \nu_j <_{lex} \nu_k \wedge \nu_l$.

Conversely, we need to show that $\mathrm{cl}(\bar{\eta}) \sim_{str} \mathrm{cl}(\bar{\nu})$ implies $\bar{\eta} \sim_\delta \bar{\nu}$. But this implication is trivial because $\mathrm{cl}(\bar{\eta})$, $\mathrm{cl}(\bar{\nu})$ are meet-closures of $\bar{\eta}$, $\bar{\nu}$, and $\mathcal{L}_{str}$ contains $\trianglelefteq$ which defines Δ. □

Lemma 6.9. *Suppose* $\kappa > \lambda \geq \omega$ *and* λ *is a limit ordinal. If* $\langle a_\eta \rangle_{\eta \in \mathbb{Q}^{<\kappa}}$ *is str-indiscernible, then* $\langle a_\eta \rangle_{\eta \in \mathbb{Q}^\lambda}$ *is* δ*-indiscernible.*

Proof. We can show this by the same argument which is used in Lemma 6.8. □

Remark 6.10. Let $\kappa \geq \omega$. The str-modeling properties for $\omega^{<\kappa}$, $\mathbb{Q}^{<\omega}$, and $\mathbb{Q}^{<\kappa}$ hold by Fact 6.4 ([13, Theorem 3.12]) since $\mathrm{age}(\omega^{<\kappa}) = \mathrm{age}(\mathbb{Q}^{<\omega}) = \mathrm{age}(\mathbb{Q}^{<\kappa}) = \mathrm{age}(\omega^{<\omega})$.

Lemma 6.11. ω^ω *has the* δ*-modeling property.*

Proof. Assume that $\langle a_\eta \rangle_{\eta \in \omega^\omega}$ is given. We want to find δ-indiscernible $\langle b_\eta \rangle_{\eta \in \omega^\omega}$ which is δ-based on $\langle a_\eta \rangle_{\eta \in \omega^\omega}$.

First we complete $\langle a_\eta \rangle_{\eta \in \omega^{\leq \omega}}$ from $\langle a_\eta \rangle_{\eta \in \omega^\omega}$ by defining $a_\eta = a_{\eta^\frown 0^\omega}$ for each $\eta \in \omega^{<\omega}$. By Remark 6.10, we can apply the str-modeling property on $\langle a_\eta \rangle_{\eta \in \omega^{\leq \omega}}$ and obtain a str-indiscernible tree $\langle b_\eta \rangle_{\eta \in \omega^{\leq \omega}}$ which is str-based on $\langle a_\eta \rangle_{\eta \in \omega^{\leq \omega}}$.

Now we show that $\langle b_\eta \rangle_{\eta \in \omega^\omega}$ is δ-indiscernible and δ-based on $\langle a_\eta \rangle_{\eta \in \omega^\omega}$. By Lemma 6.8, $\langle b_\eta \rangle_{\eta \in \omega^\omega}$ is δ-indiscernible. Choose any $\eta_0, ..., \eta_n \in \omega^\omega$ and a finite set of formulas, say Φ . Then there exist $\nu_0, ..., \nu_n \in \omega^{\leq \omega}$ such that $\mathrm{cl}(\eta_0, ..., \eta_n) \sim_{str} \mathrm{cl}(\nu_0, ..., \nu_n)$ and $\bar{b}_{\mathrm{cl}(\bar{\eta})} \equiv \bar{a}_{\mathrm{cl}(\bar{\nu})}$. In particular, we have

$$b_{\eta_0} ... b_{\eta_n} \equiv_\Phi a_{\nu_0} ... a_{\nu_n} = a_{\nu'_0} ... a_{\nu'_n},$$

where $\nu'_i = \nu_i$ if $\nu_i \in \omega^\omega$, and $\nu'_i = \nu_i{}^\frown 0^\omega$ if $\nu_i \in \omega^{<\omega}$. Since $\mathrm{cl}(\bar{\eta}) \sim_{str} \mathrm{cl}(\bar{\nu}) \sim_{str} \mathrm{cl}(\bar{\nu'})$, we have $\bar{\eta} \sim_\delta \bar{\nu'}$. Thus $\langle b_\eta \rangle_{\eta \in \omega^\omega}$ is δ-based on $\langle a_\eta \rangle_{\eta \in \omega^\omega}$. □

Lemma 6.12. *Let λ be a limit ordinal. Then ω^λ and $\mathbb{Q}^\lambda$ have the δ-modeling property and age(ω^λ), age$(\mathbb{Q}^\lambda)$ are Ramsey classes with respect to $\mathcal{L}_\delta$. Furthermore, for any dense universal antichain X in $\mathbb{Q}^{<\kappa}$ for $\kappa > \omega$, X has the δ-modeling property and age(X) is a Ramsey classes.*

Proof. By Fact 6.4 ([13, Theorem 3.12]), age(ω^ω) is a Ramsey class with respect to $\mathcal{L}_\delta$ since it has δ-modeling property. And age$(\mathbb{Q}^\lambda)$ and age(ω^λ) are Ramsey classes so that $\mathbb{Q}^\lambda$ and ω^λ have δ-modeling property since age$(\mathbb{Q}^\lambda)$ = age(ω^λ) = age(ω^ω). For a dense universal antichain X in $\mathbb{Q}^{<\kappa}$, it is clear that age(X) = age(ω^ω). Thus it has δ-modeling property and is a Ramsey class. □

Now we are ready to prove the main statement of this section.

Definition 6.13. Suppose $\langle a_\eta \rangle_{\eta \in \mathbb{Q}^{<\kappa}}$ is str-indiscernible over a set of parameters D and X is an antichain in $\mathbb{Q}^{<\kappa}$. Let $\Sigma(z)$ be a partial type over $D\langle a_\eta \rangle_{\eta \in X}$. If $\varphi(z, \bar{a}_{\bar{\eta}}, \bar{d}) \in$

$\Sigma(z)$ and $\bar{\eta} \sim_\delta \bar{\nu}$ always implies $\varphi(z, \bar{a}_{\bar{\nu}}, \bar{d}) \in \Sigma(z)$ for each $\varphi(z, \bar{y}, \bar{d}) \in \mathcal{L}(D)$ and $\bar{\eta}, \bar{\nu} \in X$, then we say $\Sigma(z)$ is *closed under* $\sim_\delta$ *in* $\langle a_\eta \rangle_{\eta \in X}$ *over* D.

Lemma 6.14. *Suppose* $\omega \leq \lambda < \kappa$ *and* λ *is a limit ordinal and* $\langle a_\eta \rangle_{\eta \in \mathbb{Q}^{<\kappa}}$ *is str-indiscernible over a set of parameters* D*. Let* X *be an endless dense universal anichain in* $\mathbb{Q}^{<\kappa}$*. Let* $\Sigma(z)$ *be a consistent partial type over* $D\langle a_\eta \rangle_{\eta \in X}$ *which is closed under* $\sim_\delta$ *in* $\langle a_\eta \rangle_{\eta \in X}$ *over* D*. Then there exists* b *such that:*

(i) $b \models \Sigma(z)$*,*

(ii) $\langle a_\eta \rangle_{\eta \in X}$ *is* δ*-indiscernible over* bD*.*

Proof. Let $\Gamma(z)$ be

$$\{\psi(a_{\eta_0}, ..., a_{\eta_n}, z) \leftrightarrow \psi(a_{\nu_0}, ..., a_{\nu_n}, z) : \psi \in \mathcal{L}(D), \langle \eta_0, ..., \eta_n \rangle \sim_\delta \langle \nu_0, ..., \nu_n \rangle \subseteq X\}.$$

We show that $\Sigma(z) \cup \Gamma(z)$ is consistent. Choose any $c \models \Sigma(z)$. Let X_0 and Γ_0 be a finite subset of X and Γ respectively. By compactness, it is enough to show that $\Sigma_{\restriction D\langle a_\eta \rangle_{\eta \in X_0}} \cup \Gamma_0$ is consistent. Γ_0 is of the form

$$\begin{gathered}\{\psi_0(a_{\eta_0^0}, ..., a_{\eta_{n_0}^0}, x) \leftrightarrow \psi_0(a_{\nu_0^0}, ..., a_{\nu_{n_0}^0}, x), \\ \cdots, \\ \psi_m(a_{\eta_0^m}, ..., a_{\eta_{nm}^m}, x) \leftrightarrow \psi_m(a_{\nu_0^m}, ..., a_{\nu_{nm}^m}, x)\},\end{gathered}$$

where $\bar{\eta}^0 \sim_\delta \bar{\nu}^0, ..., \bar{\eta}^m \sim_\delta \bar{\nu}^m$, where $\bar{\eta}^i = \langle \eta_0^i, ..., \eta_{n_i}^i \rangle$, and $\bar{\nu}^i = \langle \nu_0^i, ..., \nu_{n_i}^i \rangle$ for each $i \leq m$. Since X is endless and dense, we may assume $n_0 = ... = n_m = n$ for some $n \in \omega$ and $\bar{\eta}^0 \sim_\delta ... \sim_\delta \bar{\eta}^m \sim_\delta \bar{\nu}^0 \sim_\delta ... \sim_\delta \bar{\nu}^m$. Let $A = \bar{\eta}^0$, $B = X_0 \cup \bigcup_{i \leq m} (\bar{\eta}^i \cup \bar{\nu}^i)$, and $k = 2^{m+1}$. Since age(X) is a Ramsey class, there exists finite subset C of X such that $C \to (B)^A_k$. Let f be a map from $\binom{C}{A}$ to 2^{m+1} which is defined by

$$f(\xi_0, ..., \xi_n)(i) = \begin{cases} 0 & \text{if } \models \psi_i(a_{\xi_0}, ..., a_{\xi_n}, c) \\ 1 & \text{if } \not\models \psi_i(a_{\xi_0}, ..., a_{\xi_n}, c) \end{cases}$$

for each $i \leq m$. There exists $B' \subseteq C$ with $B' \sim_\delta B$ and $d \in 2^{m+1}$ such that $f(A') = d$ for all $A' \subseteq B'$ with $A' \sim_\delta A$. By Lemma 6.9 $\langle a_\eta \rangle_{\eta \in X}$ is δ-indiscernible over D. Thus there exists $\sigma \in Aut(\mathbb{M}/D)$ which maps $\langle a_\eta \rangle_{\eta \in B'}$ to $\langle a_\eta \rangle_{\eta \in B}$. For that σ, we have $\sigma(c) \models \Sigma_{\restriction D \langle a_\eta \rangle_{\eta \in X_0}} \cup \Gamma_0$. □

6.2 Obtaining a witness of ATP in a single free variable

As we mentioned at the beginning of this chapter, we show that every ATP theory is witnessed by a formula in a single free variable. To do this we first prove two modified lemmas of the path-collapse lemma ([3, Lemma 4.6]) for the purpose of dealing with antichains and ATP. They will be called antichain-collapse lemmas. The shapes of the lemmas will be made to reflect the construction of antichain trees. The following remark will describe what we mean by "to reflect the construction of antichain trees", although roughly.

Remark 6.15. As in Chapter 5, α_n denotes the number of all maximal antichains in $2^{<n}$. Then $\alpha_0 = 0$ and $\alpha_{n+1} = \alpha_n^2 + 1$ for each $n \in \omega$.

Proof. Let $\{X_i\}_{i \in \alpha_n}$ be the set of all maximal antichains in $2^{<n}$. Then

$$\{(\langle 0 \rangle^\frown X_i) \cup (\langle 1 \rangle^\frown X_j) : i, j < \alpha_n\} \cup \{\emptyset\}$$

is the set of all maximal antichains in $2^{<n+1}$. Thus $\alpha_{n+1} = \alpha_n^2 + 1$ for each $n \in \omega$. □

Note that to obtain all maximal antichains in $2^{<n+1}$, we first take the product of two copies of all maximal antichains in $2^{<n}$. This process is reflected by "$\{(\langle 0 \rangle^\frown X_i) \cup (\langle 1 \rangle^\frown X_j) : i, j < \alpha_n\}$" and "$\alpha_n^2$" in Remark 6.15. Secondly we add one more maximal antichain $\{\emptyset\}$ which is located below all antichains constructed in the first step,

to $\{(\langle 0\rangle^\frown X_i) \cup (\langle 1\rangle^\frown X_j) : i,j < \alpha_n\}$, and this completes the set of all maximal antichains in $2^{<n+1}$. This second process is reflected by "$\cup\{\emptyset\}$" and "$+1$".

To continue, we introduce the notion of collapsible family of antichains.

Definition 6.16. Let κ be an infinite cardinal and $X_0, ..., X_n \subseteq \mathbb{Q}^{<\kappa}$ be endless dense universal antichains with $|X_0| = ... = |X_n|$ and $X_0 \sim_{str} ... \sim_{str} X_n$. Let us consider the following condition.

($*$) For any set $C \subseteq \mathbb{M}$, $b \in \mathbb{M}$, tree indexed set $\langle a_\eta\rangle_{\eta\in\mathbb{Q}^{<\kappa}}$ which is str-indiscernible over C, and $i \leq n$, if $\langle a_\eta\rangle_{\eta\in X_i}$ is δ-indiscernible over bC, then $\bigcup_{j\leq n} p(y, \langle a_\eta\rangle_{\eta\in X_j})$ is consistent where $p(y,\bar{z}) = \text{tp}(b, \langle a_\eta\rangle_{\eta\in X_i}/C)$ or there is a formula with free variable y which witnesses ATP.

If $X_0, ..., X_n$ satisfies ($*$), then we say they are *collapsible*. By a *collapsible family*, we mean a set of endless dense universal antichains which are collapsible.

Remark 6.17. Let $X_0, ..., X_n$ are collapsible and $X_0...X_n \sim_{str} X'_0...X'_n$. Then $X'_0, ..., X'_n$ are also collapsible.

Now we prove two antichain-collapse lemmas. The first antichain-collapse lemma claims that the product of two collapsible families is also collapsible.

Lemma 6.18. (1st Antichain Collapse) *Let κ be a sufficiently large cardinal, and $\{X_0$, ..., $X_n\}$ be a collapsible family in $\mathbb{Q}^{<\kappa}$. Then for any $\nu, \xi \in \mathbb{Q}^{<\kappa}$ with $\nu \perp \xi$ and $\nu <_{lex} \xi$,*

$$\{X^\nu_i \cup Y \cup X^\xi_j : i,j \leq n\}$$

is a collapsible family, where $X^\nu_i = \nu^\frown X_i$, $X^\xi_j = \xi^\frown X_j$ for each $i,j \leq n$, and

$$Y = \{\nu \wedge \xi^\frown\langle s\rangle^\frown\eta : \nu(l(\nu\wedge\xi)+1) < s < \xi(l(\nu\wedge\xi)+1), \eta \in \mathbb{Q}^\omega\}.$$

Proof. Clearly $X_i^\nu \cup Y \cup X_j^\xi$ is an endless dense universal antichain for each $i, j \leq n$, and $X_i^\nu \cup Y \cup X_j^\xi \sim_{str} X_{i'}^\nu \cup Y \cup X_{j'}^\xi$ for all $i, i', j, j' \leq n$. Suppose $C \subseteq \mathbb{M}$, $b \in \mathbb{M}$, and $\langle a_\eta \rangle_{\eta \in \mathbb{Q}^{<\kappa}}$ is a str-indiscernible tree over C. Assume $\langle a_\eta \rangle_{\eta \in X_{i_0}^\nu \cup Y \cup X_{j_0}^\xi}$ is δ-indiscernible over bC for some $i_0, j_0 \leq n$. Let

$$p(y, \bar{z}, \bar{u}, \bar{w}) = \text{tp}(b, \langle a_\eta \rangle_{\eta \in X_{i_0}^\nu}, \langle a_\eta \rangle_{\eta \in Y}, \langle a_\eta \rangle_{\eta \in X_{j_0}^\xi} / C).$$

We first show that $\bigcup_{j \leq n} p(y, \langle a_\eta \rangle_{\eta \in X_{i_0}^\nu}, \langle a_\eta \rangle_{\eta \in Y}, \langle a_\eta \rangle_{\eta \in X_j^\xi})$ is consistent or there is a formula with free variable y which witnesses ATP. Since $\langle a_\eta \rangle_{\eta \in X_{i_0}^\nu \cup Y \cup X_{j_0}^\xi}$ is δ-indiscernible over bC, it is true that $\langle a_\eta \rangle_{\eta \in X_{j_0}^\xi}$ is δ-indiscernible over $bC\langle a_\eta \rangle_{\eta \in X_{i_0}^\nu \cup Y}$. Note that $\langle a_\eta \rangle_{\eta \in \xi^\frown \mathbb{Q}^{<\kappa}}$ is str-indiscernible over $C\langle a_\eta \rangle_{\eta \in X_{i_0}^\nu \cup Y}$. Let

$$q(y, \bar{w}) = \text{tp}(b, \langle a_\eta \rangle_{\eta \in X_{j_0}^\xi} / C\langle a_\eta \rangle_{\eta \in X_{i_0}^\nu \cup Y}).$$

Then $\bigcup_{j \leq n} q(y, \langle a_\eta \rangle_{\eta \in X_j^\xi})$ is consistent or there is a formula with free variable y which witnesses ATP. Note that

$$p(y, \langle a_\eta \rangle_{\eta \in X_{i_0}^\nu}, \langle a_\eta \rangle_{\eta \in Y}, \langle a_\eta \rangle_{\eta \in X_j^\xi}) \subseteq q(y, \langle a_\eta \rangle_{\eta \in X_j^\xi})$$

for each $j \leq n$. Thus $\bigcup_{j \leq n} p(y, \langle a_\eta \rangle_{\eta \in X_{i_0}^\nu}, \langle a_\eta \rangle_{\eta \in Y}, \langle a_\eta \rangle_{\eta \in X_j^\xi})$ is consistent or there is a formula with free variable y which witnesses ATP.

Now we show that $\bigcup_{i \leq n} \bigcup_{j \leq n} p(y, \langle a_\eta \rangle_{\eta \in X_i^\nu}, \langle a_\eta \rangle_{\eta \in Y}, \langle a_\eta \rangle_{\eta \in X_j^\xi})$ is consistent or there is a formula with free variable y which witnesses ATP. Assume that there is no formula with free variable y which witnesses ATP. Then $\bigcup_{j \leq n} q(y, \langle a_\eta \rangle_{\eta \in X_j^\xi})$ is consistent, as we observed above. Note that $\langle a_\eta \rangle_{\eta \in X_{i_0}^\nu}$ is δ-indiscernible over $bC\langle a_\eta \rangle_{\eta \in X_{j_0}^\xi \cup Y}$. Thus $\bigcup_{j \leq n} q(y, \langle a_\eta \rangle_{\eta \in X_j^\xi})$ is closed under $\sim_\delta$ in $\langle a_\eta \rangle_{\eta \in X_{i_0}^\nu}$ over $C\langle a_\eta \rangle_{\eta \in Y \cup X_{\leq n}^\xi}$. Since $\langle a_\eta \rangle_{\eta \in \nu^\frown \mathbb{Q}^{<\kappa}}$ is str-indiscernible over $C\langle a_\eta \rangle_{\eta \in Y \cup X_{\leq n}^\xi}$, we can apply Lemma 6.14 to obtain b' such that

$$\text{tp}(b'/C\langle a_\eta\rangle_{\eta\in X^\nu_{i_0}\cup Y\cup X^\xi_{\leq n}}) \supseteq \bigcup_{j\leq n} q(y, \langle a_\eta\rangle_{\eta\in X^\xi_j}),$$

and $\langle a_\eta\rangle_{\eta\in X^\nu_{i_0}}$ is δ-indiscernible over $b'C\langle a_\eta\rangle_{\eta\in Y\cup X^\xi_{\leq n}}$. Let

$$r(y,\bar{z}) = \text{tp}(b', \langle a_\eta\rangle_{\eta\in X^\nu_{i_0}}/C\langle a_\eta\rangle_{\eta\in Y\cup X^\xi_{j\leq n}}).$$

Then $\bigcup_{i\leq n} r(y, \langle a_\eta\rangle_{\eta\in X^\nu_i})$ is consistent since we assume there is no ATP witness with free variable y. It is clear that

$$\bigcup_{j\leq n} p(y, \langle a_\eta\rangle_{\eta\in X^\nu_i}, \langle a_\eta\rangle_{\eta\in Y}, \langle a_\eta\rangle_{\eta\in X^\xi_j}) \subseteq r(y, \langle a_\eta\rangle_{\eta\in X^\nu_i})$$

for each $i \leq n$. Thus $\bigcup_{i\leq n}\bigcup_{j\leq n} p(y, \langle a_\eta\rangle_{\eta\in X^\nu_i}, \langle a_\eta\rangle_{\eta\in Y}, \langle a_\eta\rangle_{\eta\in X^\xi_j})$ is consistent. This completes the proof. □

More generally, we can prove the following statement by using the same argument in the first antichain-collapse lemma.

Notation 6.19. For $X \subseteq \mathbb{Q}^{<\kappa}$, let $\bigwedge X = \max\{\xi \in \mathbb{Q}^\kappa : \forall\eta \in X(\xi \trianglelefteq \eta)\}$. For $X, Y \subseteq \mathbb{Q}^{<\kappa}$, we write $X <_{lex} Y$ if $\eta <_{lex} \nu$ for all $\eta \in X$, and $\nu \in Y$.

Lemma 6.20. *Let κ be a sufficiently large cardinal. Let $\{X_0, ..., X_n\}$ and $\{Y_0, ..., Y_m\}$ be collapsible families in $\mathbb{Q}^{<\kappa}$. Let $\nu = \bigwedge\bigcup X_i$ and $\xi = \bigwedge\bigcup Y_j$. Suppose $X_i <_{lex} Y_j$ for $i \leq n$, $j \leq m$ with $\nu \perp \xi$ so that $\nu <_{lex} \xi$. Then*

$$\{X_i \cup Z \cup Y_j : i, j \leq n\}$$

is a collapsible family, where

$$Z = \{\nu \wedge \xi^\frown\langle s\rangle^\frown\eta : \nu(l(\nu\wedge\xi)+1) < s < \xi(l(\nu\wedge\xi)+1), \eta \in \mathbb{Q}^\omega\}.$$

Then we get the following proposition by applying the first antichain-collapse lemma repeatedly.

Proposition 6.21. *Let $\{X_0, ..., X_n\}$ be a collapsible family in $\mathbb{Q}^{<\kappa}$. Let $X = \{\xi_0 <_{lex} ... <_{lex} \xi_m\}$ be a finite antichain in $\mathbb{Q}^{<\kappa}$. Then there is an antichain $Y \subseteq \mathbb{Q}^{<\kappa}$ such that*

$$\{Y \cup \bigcup_{k\leq m} X_i^{\xi_k} : i \leq n\}$$

is a collapsible family, where $X_i^{\xi_k} = \xi_k{}^\frown X_i$ for each $i \leq n$ and $k \leq m$.

Now we prove the second antichain-collapse lemma which says that every collapsible family $\mathcal{F}$ can be extended to the collapsible family $\mathcal{F}' \cup \{X\}$ where $\mathcal{F}' \sim_{str} \mathcal{F}$ and X is an antichain located below $\mathcal{F}'$.

Lemma 6.22. (2nd Antichain Collapse) *Let κ be a sufficiently large cardinal, and $\{X'_0,$ ..., $X'_n\}$ be a collapsible family in $\mathbb{Q}^{<\kappa}$. Then there is a collapsible family $\{X_0,$ ..., $X_{n+1}\}$ in $\mathbb{Q}^{<\kappa}$ which satisfies that there are $\chi \in X_{n+1}$ and $\chi' \trianglerighteq \chi$ such that $X_0 = \chi'{}^\frown X'_0, ..., X_n = \chi'{}^\frown X'_n$.*

Proof. Choose any $\nu' \in X'_0$. Then there is $\nu \in \mathbb{Q}^{<\kappa}$ such that $\nu' \trianglelefteq \nu$ and $\nu \not\trianglelefteq \eta$ for all $\eta \in X'_i$, for all $i \leq n$. Let $X_i = \nu{}^\frown\langle 0\rangle{}^\frown X'_i$ for each $i \leq n$ and $X_{n+1} = X'_0$. Then $X_0, ..., X_{n+1}$ are endless dense universal antichains with $|X_0| = ... = |X_{n+1}|$ and $X_0 \sim_{str} ... \sim_{str} X_{n+1}$.

We show $X_0, ..., X_{n+1}$ are collapsible. Suppose $C \subseteq \mathbb{M}$, $b \in \mathbb{M}$, and $\langle a_\eta\rangle_{\eta\in\mathbb{Q}^{<\kappa}}$ is a str-indiscernible tree over C. First we assume $\langle a_\eta\rangle_{\eta\in X_{n+1}}$ is δ-indiscernible over bC and show $\bigcup_{i\leq n+1} p(y, \langle a_\eta\rangle_{\eta\in X_i})$ is consistent or there is a formula with free variable y which witnesses ATP, where $p(y, \bar{z}) = \text{tp}(b, \langle a_\eta\rangle_{\eta\in X_{n+1}}/C)$. Let us assume $\bigcup_{i\leq n+1} p(y, \langle a_\eta\rangle_{\eta\in X_i})$ is inconsistent. Since $X_{n+1} \sim_{str} X_0$, there exists b' such that $\text{tp}(b', \langle a_\eta\rangle_{\eta\in X_0}/C) = \text{tp}(b, \langle a_\eta\rangle_{\eta\in X_{n+1}}/C)$. Hence, if $\bigcup_{i\leq n} p(y, \langle a_\eta\rangle_{\eta\in X_i})$ is inconsistent, then there is a formula with free variable y which witnesses ATP because $\langle a_\eta\rangle_{\eta\in X_0}$

is δ-indiscernible over $b'C$, and $X_0, ..., X_n$ are collapsible. Therefore we suppose that $\bigcup_{i\leq n} p(y, \langle a_\eta\rangle_{\eta\in X_i})$ is consistent. Then there exists a formula $\varphi(y, \bar{z})$ in $p(y, \bar{z})$ such that $\bigwedge_{i\leq n+1}\varphi(y, \bar{a}_{\bar{\eta}^i})$ is not consistent and $\bigwedge_{i\leq n}\varphi(y, \bar{a}_{\bar{\eta}^i})$ is consistent, where $\bar{\eta}^i = \langle\eta^i_0, ..., \eta^i_k\rangle \in X_i$ for each $i \leq n+1$ and $\bar{\eta}^i \sim_{str} \bar{\eta}^j$ for all $i, j \leq n+1$. For each $i \leq n$ and $j \leq k$, there exists $\xi^i_j \in X'_i$ such that $\nu^\frown\langle 0\rangle^\frown\xi^i_j = \eta^i_j$. We may assume $\xi^0_j = \eta^{n+1}_j$ for all $j \leq k$. We construct a binary tree $(\bar{a}^\varepsilon)_{\varepsilon\in 2^{<\omega}}$ by $\bar{a}^\varepsilon = \langle\bar{a}_{\chi_\varepsilon{}^\frown\bar{\xi}^0}, ..., \bar{a}_{\chi_\varepsilon{}^\frown\bar{\xi}^n}\rangle$ where

$$\begin{aligned}\chi_\emptyset &= \emptyset,\\ \chi_{\varepsilon^\frown\langle 0\rangle} &= \chi_\varepsilon{}^\frown\nu^\frown\langle 0\rangle,\\ \chi_{\varepsilon^\frown\langle 1\rangle} &= \chi_\varepsilon{}^\frown\nu^\frown\langle 1\rangle.\end{aligned}$$

Let $\psi(y, \bar{z}_0, ..., \bar{z}_n) = \bigwedge_{i\leq n}\varphi(y, \bar{z}_i)$. Clearly $\{\psi(y, \bar{a}^\varepsilon), \psi(y, \bar{a}^{\varepsilon'})\}$ is inconsistent for all $\varepsilon \trianglelefteq\!\!\!\!\!\!\neq \varepsilon'$. To show ψ witnesses ATP with $(\bar{a}^\varepsilon)_{\varepsilon\in 2^{<\omega}}$, it is enough to show that $\{\psi(y, \bar{a}^\varepsilon)\}_{\varepsilon\in 2^m}$ is consistent for each $m \in \omega$. By Proposition 6.21, we can find Y such that

$$\{Y \cup \bigcup_{\varepsilon\in 2^m}\chi_\varepsilon{}^\frown X'_i : i \leq n\}$$

is a collapsible family. Note that $\bigwedge_{\varepsilon\in 2^m}\varphi(y, \bar{a}_{\chi_\varepsilon{}^\frown\bar{\xi}^0})$ is consistent since

$$\varphi(y, \bar{z}) \in \text{tp}(b, \langle a_\eta\rangle_{\eta\in X_{n+1}}/C),$$

$\langle a_\eta\rangle_{\eta\in X_{n+1}}$ is δ-indiscernible over bC, $\langle\chi_\varepsilon{}^\frown\bar{\xi}^0\rangle_{\varepsilon\in 2^m}$ is a finite antichain, and X_{n+1} is universal. Let $\Sigma(y)$ be

$$\{\bigwedge_{\varepsilon\in 2^m}\varphi(y, \bar{a}_{\chi'_\varepsilon}) : \langle\chi_\varepsilon{}^\frown\bar{\xi}^0\rangle_{\varepsilon\in 2^m} \sim_\delta \langle\chi'_\varepsilon\rangle_{\varepsilon\in 2^m} \in Y \cup \bigcup_{\varepsilon\in 2^m}\chi_\varepsilon{}^\frown X'_0\}.$$

Then by the universality of X_{n+1} again, $\Sigma(y)$ is consistent. Clearly $\Sigma(y)$ is closed under $\sim_\delta$ in $\langle a_\eta\rangle_{\eta\in Y\cup\bigcup_{\varepsilon\in 2^m}\chi_\varepsilon{}^\frown X'_0}$ over C. Thus we can apply Lemma 6.14 to obtain

$b'' \models \bigwedge_{\varepsilon\in 2^m} \varphi(y, \bar{a}_{\chi_\varepsilon{}^\frown \bar{\xi}^0})$ such that $Y \cup \bigcup_{\varepsilon\in 2^m} \chi_\varepsilon{}^\frown X'_0$ is δ-indiscernible over $b''C$. Thus $\bigcup_{i\leq n} q(y, \langle a_\eta\rangle_{\eta\in Y\cup\bigcup_{\varepsilon\in 2^m}\chi_\varepsilon{}^\frown X'_i})$ is consistent where

$$q(y, \bar{w}) = \mathrm{tp}(b'', \langle a_\eta\rangle_{\eta\in Y\cup\bigcup_{\varepsilon\in 2^m}\chi_\varepsilon{}^\frown X'_0}).$$

In particular, $\{\psi(y, \bar{a}^\varepsilon)\}_{\varepsilon\in 2^m}$ is consistent. Thus there is a formula with free variable y which witnesses ATP. This is a contradiction.

Now we assume $\langle a_\eta\rangle_{\eta\in X_{i_0}}$ is δ-indiscernible over bC for some $i_0 \leq n$, and show that $\bigcup_{i\leq n+1} p(y, \langle a_\eta\rangle_{\eta\in X_i})$ is consistent or there is a formula with free variable y which witnesses ATP, where $p(y, \bar{z}) = \mathrm{tp}(b, \langle a_\eta\rangle_{\eta\in X_{i_0}})$. Since $X_{i_0} \sim_{str} X_{n+1}$, there exists b' such that $\mathrm{tp}(b, \langle a_\eta\rangle_{\eta\in X_{i_0}}) = \mathrm{tp}(b', \langle a_\eta\rangle_{\eta\in X_{n+1}})$ and $\langle a_\eta\rangle_{\eta\in X_{n+1}}$ is δ-indiscernible over $b'C$. As we observed above, $\bigcup_{i\leq n+1} p(y, \langle a_\eta\rangle_{\eta\in X_i})$ is consistent or there is a formula with free variable y which witnesses ATP. □

Now we are ready to prove the main theorem of this chapter. Recall that α_m is the number of all maximal antichains in $2^{<m}$.

Theorem 6.23. *If there exists a witness of ATP, then there exists a witness of ATP in a single free variable.*

Proof. Let $\varphi(x, y; z)$ be a witness of ATP with str-indiscernible $\langle a_\eta\rangle_{\eta\in\mathbb{Q}^{<\kappa}}$, where $|a_\eta| = |z|$ for all $\eta \in \mathbb{Q}^{<\kappa}$, $|y| = 1$, and κ is sufficiently large. First we construct a collapsible family $\{X_{m,i}\}_{i<\alpha_m}$ and find $f_m : 2^{<m} \to \mathbb{Q}^{<\kappa}$ for each $m < \omega\backslash\{0\}$ which satisfy

(i) for each maximal antichain $Y \subseteq 2^{<m}$, there exists a unique $i < \alpha_m$ such that $f_m(Y) \subseteq X_{m,i}$,

(ii) for any distinct maximal antichains $Y, Y' \subseteq 2^{<m}$, if $f_m(Y) \subseteq X_{m,i}$ and $f_m(Y') \subseteq X_{m,j}$, then $i \neq j$,

(iii) f_m is a str-embedding, which means $\bar{\eta} \sim_{str} f_m(\bar{\eta})$ for all $\bar{\eta} \in 2^{<m}$.

Let $X_{1,0} = \mathbb{Q}^\omega$ and $f_1(\emptyset) = 0^\omega \in \mathbb{Q}^\omega$. Then $\{X_{1,i}\}_{i<\alpha_1}$ and f_1 satisfy the conditions. Let us assume we have constructed $\{X_{m,i}\}_{i<\alpha_m}$ and $f_m : 2^{<m} \to \mathbb{Q}^{<\kappa}$. By the first antichain-collapse lemma,

$$\Gamma = \{X^{\langle 0\rangle}_{m,i} \cup Y \cup X^{\langle 1\rangle}_{m,j} : i, j < \alpha_m\}$$

is a collapsible family, where $X^{\langle 0\rangle}_{m,i} = \langle 0\rangle^\frown X_{m,i}$, $X^{\langle 1\rangle}_{m,j} = \langle 1\rangle^\frown X_{m,j}$ for each $i, j \leq \alpha_m$, and

$$Y = \{\langle s\rangle^\frown \eta : 0 < s < 1, s \in \mathbb{Q}, \eta \in \mathbb{Q}^\omega\}.$$

Let $\{X'_{m,i}\}_{i<\alpha_m^2}$ be an enumeration of Γ. By the second antichain-collapse lemma, there is a collapsible family $\{X_{m+1,i}\}_{i<\alpha_{m+1}}$, which satisfies $X_{m+1,i} = \chi'^\frown X'_{m,i}$ for all $i < \alpha_m^2$, for some $\chi \in X_{m+1,\alpha_m^2}$, and $\chi' \trianglerighteq \chi$. Let f_{m+1} be a map from $2^{<m+1}$ to $\mathbb{Q}^{<\kappa}$ which is defined by

$$f_{m+1}(\eta) = \begin{cases} \chi & \text{if } \eta = \emptyset \\ \chi'^\frown\langle 0\rangle^\frown f_m(\eta') & \text{if } \eta = \langle 0\rangle^\frown\eta' \\ \chi'^\frown\langle 1\rangle^\frown f_m(\eta') & \text{if } \eta = \langle 1\rangle^\frown\eta'. \end{cases}$$

Then $\{X_{m+1,i}\}_{i<\alpha_{m+1}}$ and f_{m+1} satisfy (i),(ii), and (iii).

To get a contradiction, let us assume that there is no witness of ATP in a single free variable. We show that for each $m < \omega$, there exists b_m with $|b_m| = |y|$ such that $\varphi(x; y, z)$ satisfies the conditions of ATP with $\langle b_m a_{f_m(\eta)}\rangle_{\eta\in 2^{<m}}$. Choose any $i_0 < \alpha_m$. Then there exists b' such that $\{\varphi(x, b', a_\eta)\}_{\eta\in X_{m,i_0}}$ is consistent and $\langle a_\eta\rangle_{\eta\in X_{m,i_0}}$ is δ-indiscernible over b', by Lemma 6.14. Since $\{X_{m,i}\}_{i<\alpha_m}$ is collapsible, $\bigcup_{i<\alpha_m} p(y, \langle a_\eta\rangle_{\eta\in X_{m,i}})$ is consistent where $p(y, \bar{z}) = \text{tp}(b', \langle a_\eta\rangle_{\eta\in X_{m,i_0}})$. Choose any $b_m \models \bigcup_{i<\alpha_m} p(y, \langle a_\eta\rangle_{\eta\in X_{m,i}})$. Then $\varphi(x; y, z)$ witnesses ATP with $\langle b_m a_{f_m(\eta)}\rangle_{\eta\in 2^{<m}}$.

By compactness, there exist b and str-embedding $f : 2^{<\omega} \to \mathbb{Q}^{<\kappa}$ such that $\varphi(x; y, z)$ witnesses ATP with $\langle ba_{f(\eta)} \rangle_{\eta \in 2^{<\omega}}$. By repeating this process we can reduce the number of free variables in the witness of ATP and finally we obtain a witness of ATP in a single free variable. But this is a contradiction with what we assumed at the beginning of this paragraph. Thus there exists a witness of ATP in a single free variable. □

Corollary 6.24. *If there exists a witness of SSOP$_1$, then there exists a witness of SSOP$_1$ in a single free variable.*

Proof. By Proposition 4.4 and 4.6, we can show that if $\varphi(x; y)$ witnesses ATP in a single free variable x, then it witnesses SSOP$_1$ in the same free variable, and vice versa. □

Therefore, if we want to check whether a theory is NATP (NSSOP$_1$), it is enough to show that there is no formula in a single free variable which witnesses ATP (SSOP$_1$). Furthermore, if the theory has quantifier elimination, then the verification will be much easier by the following observation.

Proposition 6.25. *Witnessing NATP is closed under disjunction. That is, if $\varphi \vee \psi$ witnesses ATP, then φ witnesses ATP or ψ witnesses ATP.*

Proof. Suppose $\varphi(x, y) \vee \psi(x, z)$ witnesses ATP with str-indiscernible $\langle a_\eta b_\eta \rangle_{\eta \in \mathbb{Q}^{<\kappa}}$, where κ is sufficiently large. Let $c \models \{\varphi(x, a_\eta) \vee \psi(x, b_\eta)\}_{\eta \in \mathbb{Q}^\omega}$. For each $n \in \omega$, let B_n be a subset of $\mathbb{Q}^\omega$ which is str-similar to $2^n \subseteq \mathbb{Q}^{<\kappa}$. Since $\mathbb{Q}^\omega$ is a Ramsey class with respect to $\mathcal{L}_\delta$, there exists $C_n \subseteq \mathbb{Q}^\omega$ such that $C_n \to (B_n)^A_2$ for each $n \in \omega$, where $A = \{\langle 0^\omega \rangle\}$. Let $f_n : C_n \to 2$ be a coloring which is defined by

$$f_n(\eta) = \begin{cases} 0 & \text{if } \models \varphi(c, a_\eta) \\ 1 & \text{otherwise.} \end{cases}$$

Then there exist $B'_n \subseteq C_n$ and $i_n < 2$ such that $B'_n \sim_\delta B_n$ and $f_n(\eta) = i_n$ for all $\eta \in B'_n$. By the pigeon hole principle, there exist $i < 2$ and an infinite subset I of ω such that $i_n = i$ for all $n \in I$. We may assume $i = 0$.

Now we show that φ witnesses ATP with $\langle a_\eta \rangle_{\eta \in \mathbb{Q}^{<\kappa}}$. If $\eta \trianglelefteq \nu$, then clearly $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent since $\{\varphi(x, a_\eta) \vee \psi(x, b_\eta), \varphi(x, a_\nu) \vee \psi(x, b_\eta)\}$ is inconsistent. Let $X \subseteq \mathbb{Q}^{<\kappa}$ be an antichain. By compactness we may assume X is finite. Then there exists $n \in \omega$ such that there exists $X' \subseteq 2^n$ such that $X \sim_{str} X'$. Since I is infinite, we may assume $n \in I$. Since $i_n = i = 0$, the set $\{\varphi(x, a_\eta)\}_{\eta \in B'_n}$ is consistent. Since $B'_n \sim_\delta B_n \sim_{str} 2^n$, the set $\{\varphi(x, a_\eta)\}_{\eta \in 2^n}$ is consistent. In particular $\{\varphi(x, a_\eta)\}_{\eta \in X'}$ is consistent, and hence $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent. This shows that φ witnesses ATP and completes the proof. □

Corollary 6.26. *Witnessing SSOP$_1$ is also closed under disjunction.*

Hence, if a theory has quantifier elimination, then to show the theory is NATP (NSSOP$_1$) we only need to check if there is no formula in a single free variable, which is the conjunction of atomic formulas and negation of atomic formulas, witnesses ATP (SSOP$_1$).

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국문 초록

SOP_1, SOP_2, 반사슬 나무 성질

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이 논문은 두 가지 모형이론적 수형 형질 SOP_1과 SOP_2의 관계에 대한 연구이다. 연구는 먼저 새로운 종류의 비분별성인 β-비분별성을 소개하고, SOP_1을 갖는 모든 이론은 β-비분별성을 갖는 SOP_1의 증인을 갖고 있음을 보인다. 그리고 이를 이용해 모든 SOP_1 이론에 대해, 만약 그 이론이 SOP_2를 갖지 않는다면 반드시 반사슬 수형 성질 (antichain tree property, ATP)를 갖게 됨을 증명한다. ATP는 조합론적 관점에서 SOP_2와 반대되는 특징을 가지고 있으며, 연구에서는 어떤 이론이 ATP를 갖고 있으면, 그 이론은 반드시 SOP_1과 TP_2를 갖게 됨이 밝혀진다.

연구에서는 또한 새로운 수형 성질인 $SSOP_1$을 정의하고 이것이 SOP_1의 특별한 경우이며 앞서 SOP_1의 경우와 마찬가지로 어떤 이론이 $SSOP_1$을 가지면 그 이론은 반드시 β-비분별성을 갖는 $SSOP_1$의 증인을 갖게 됨을 증명한다. 그리고 어떤 이론이 $SSOP_1$를 가지게 되는 것과 ATP를 가지게 되는 것이 동치임을 보인다.

이 결과들은 어떤 이론이 SOP_1을 갖는 것이 SOP_2를 갖는 것과 동치인지를 묻는 문제를 어떤 이론이 $SSOP_1$ 혹은 ATP를 갖는 것이 SOP_2를 갖는 것과 동치인지를 묻는 문제로 한정시켜준다. 즉, 만약 어떤 이론이 SOP_2를 가짐을 그 이론이 SOP_1를 가진다는 가정으로부터 이끌어내고 싶으면, 우리는 그 이론이 SOP_1보다 더 강력한 조건인 $SSOP_1$과 ATP를 가진다고 간주해도 되는 것이다.

본 연구는 또한 ATP를 갖는 이론의 예를 구성해 보여준다. 이 이론은 ATP의 증인이 되는 양화사 없는 정식을 가지고 있으며, 동시에 어떠한 양화사 없는 정식도 SOP_2를 만족하지 않는다. 이것을 통해 우리는 SOP_1과 SOP_2가 정식 수준에서는 같은 성질이 아님을 확인하게 된다.

마지막으로 우리는 반사슬에 대한 조합론적 연산 기술들을 살펴본다. 그리고 그것의 응용으로써 어떤 이론이 ATP를 갖고 있다면, 그것은 항상 일변수 정식으로 확인될 수 있음을 증명한다.

핵심단어: 수형 성질, SOP_1, $SSOP_1$, SOP_2, 비분별 수열, 반사슬 나무 성질, 일변수 정리